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KRAICHNAN TURBULENCE VIA FINITE TIME AVERAGES

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Abstract. Relations central to Kraichnan's theory of fully developed two-dimensional turbulence are rigorously established for finite time averages. In particular, we prove that if the ratio of the averages of palinstrophy to enstrophy is large, then a large inertial range displaying an enstrophy cascade exists. Moreover, if this ratio is comparable (up to a logarithm) to the dissipation wave number (a necessary condition for turbulence), then the power law for the energy spectrum, until now derived only heuristically, is rigorously shown to provide (up to a logarithm) an upper bound for the energy spectrum. Finally we show that, deep in the dissipation range, the palinstrophy contributed by eddies smaller than a specified length decays exponentially in the corresponding wave number. The averaging times needed for these relations are bounded in terms of the generalized Grashof number, independent of the solution for which the time averages are taken. Solutions are not assumed to be on the global attractor, merely within the absorbing ball, an easily verified condition.

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INTRODUCTION

Patterns in turbulent flows are observable only after suitable averaging. This accounts for the reference to statistics in the very title of the classic treatise on turbulence by Monin and Yaglom [MY]. The Kraichnan theory for fully developed 2-D turbulence (inspired by that of Kolmogorov in 3-D) asserts that on average the behavior of eddies in such flows is determined by their length scales. In a relatively large range of scales (called the inertial range) viscous effects are negligible and enstrophy (energy in 3-D) is transferred at a nearly constant rate from one length scale to the next smaller one (a cascade). The dissipation range consists of the very small length scales where the viscosity has an annihilating effect.

The demarcation of these ranges by universal, critical length scales (or inversely, wave numbers) is one main component of these theories. Heuristic arguments by Batchelor [B53] and Kraichnan [K67] (and originally by Kolmogorov [K41] in 3-D), place the dissipation range beyond a wave number $\kappa_\eta = (\eta/\nu^3)^{1/6}$, where $\nu$ is the kinematic viscosity and $\eta$ is the average dissipation rate of enstrophy/mass (resp. $\kappa_\varepsilon = (\varepsilon/\nu^3)^{1/4}$, $\varepsilon$, of energy/mass). The theories also assert that in turbulent flows these wave numbers are large compared to those of the enstrophy (resp. energy) feeding structures; the inertial range lies in between. Concerning the dissipation range, this theory tacitly assumes that the average dissipation rate of enstrophy/mass in the eddies beyond $\kappa_\eta$ is small compared to $\eta$ (resp. energy/mass, $\kappa_\varepsilon$, $\varepsilon$).

Another main component is the distribution of energy over all scales (but with a special emphasis on the inertial range). In classical homogeneous turbulence (which serves as an informal mathematical framework for the empirical theories referred to above) the average (per unit mass) of the energy $e_{\kappa_1,\kappa_2}$ over the range of wave numbers ($\kappa_1, \kappa_2$) is considered to be an integral (with respect to wave numbers over that range) with a certain integrand called the energy spectrum $S$. Several heuristic arguments (see [B53,K67]; see also [FNS]) lead to a power law for the energy spectrum in the inertial range of the form $S(\kappa) \sim \eta^{2/3} \kappa^{-5/3}$ (resp. $\varepsilon^{2/3} \kappa^{-5/3}$ in 3-D), which in turn, is consistent with

$$e_{\kappa,2\kappa} \sim \eta^{2/3} \kappa^{-2},$$

(resp. $\varepsilon^{2/3} \kappa^{-2/3}$ in 3-D).

The rigorous mathematical framework for this paper is provided by the 2-D Navier-Stokes equations for flows which are periodic in both spatial variables. Viewed as an infinite dimensional dynamical system in the appropriate function space, all trajectories eventually enter and remain in an absorbing ball after a time which is uniform over all initial data of a given radius. Within the absorbing ball lies a finite-dimensional set called the global attractor, the largest bounded invariant set [H,T97]. Estimates for the radius of the absorbing ball are easily derived (see section 1) and sharp bounds on the dimension of the global attractor under general circumstances can be found in [CFT89]. It is shown in [FJMR] that the bound on the dimension can be made even sharper in the case of fully developed turbulence.

The ubiquitous averages in the theories outlined above can in practice be taken in two ways: over a large ensemble of flows or over a long time interval for a single initial
condition. The connection between the two approaches is mathematically rigorous, but esoteric. Indeed the time averages may not converge in the classical sense. However any Hahn-Banach extension of the notion of limit (a nonconstructive functional analysis device) applied to the time averages yields an invariant measure on the global attractor. Ensemble averages with respect to this invariant probability measure match the generalized limits of the time averages (see section 2). The advantage of this approach is that it provides a mathematical setting for rigorous results relating Kraichnan 2-D turbulence theory to the Navier-Stokes equations [FJMR, FMRT2]. It is clear that this mathematical treatment, although consistent with what is done in practice, does not explain why patterns emerge after averaging over time intervals which are finite. In this paper we prove that this basic empirical fact can be also rigorously inferred from the Navier-Stokes equations, and give estimates for the averaging times of some of the fundamental physical variables and relations. The estimates for the averaging times are uniform for all solutions in the absorbing ball.

Finite time averages offer a computational advantage over ensemble averages. Direct numerical simulation using any initial data will result in a trajectory converging to the global attractor. For systems of modest complexity alternative numerical algorithms have successfully detected individual membership in global attractors [FT1, FT2, FJ, FJKu, FJL], and constructed invariant measures on these sets [DHJR]. Yet complete determination of the global attractor, let alone that of an appropriate invariant measure on that set in the case of the Navier-Stokes equations in a turbulent regime is not practical. In contrast, the condition for the finite time average that the initial data be in the absorbing ball is easily verified. While our current rigorous estimates for the averaging time may be too large to achieve in practice, it is expected that shorter times would in fact suffice.

We reproduce here in terms of finite time averages, the main estimates in [FJMR] which were proved for ensemble averages. In particular, we show that an enstrophy cascade exists, provided both that a certain wave number, $\kappa_\sigma$ is large compared to a given upper bound $\Pi$ on the wave number of the force, and the averaging time is on the order of the generalized Grashof number $G$ (see Definition (1.15)). The wave number $\kappa_\sigma$ is defined as the square root of the ratio $\eta/\epsilon$. It is remarkable that this purely global condition on the solution can yield such a local (in wave number) result. While it is not known if this condition is achievable, we prove that if the Kraichnan theory holds, then so does $\kappa_\sigma \gg \Pi$.

Since the Kraichnan theory stipulates that $\kappa_\eta \gg \Pi$, we seek to relate $\kappa_\sigma$ to $\kappa_\eta$ as was done in [FJMR] for ensemble averages. To do so, we first prove a finite time average analogue of an estimate in [FMT93] which ensures that $\kappa_\eta/\Pi \gg 1$, if and only if $G \gg 1$. The latter is easily achieved by increasing the magnitude of the driving force. Assuming only that $G$ is large and the averaging time again on the order of $G$, we show that $\kappa_\sigma \lesssim \kappa_\eta$. If, on the other hand, the Kraichnan theory holds, we have the more precise relation

$$\kappa_\sigma \sim \kappa_\eta \left( \ln \frac{\kappa_\eta}{\Pi} \right)^{-1/2}.$$  

It is easy to show that (1.2) is equivalent to

$$\kappa_\epsilon \sim \kappa_\eta \left( \ln \frac{\kappa_\eta}{\Pi} \right)^{1/4}.$$
The results obtained in this paper readily yield (by applying a generalized Hahn-Banach limit to our finite time averages) the corresponding results in [FJMR]. Although, in fairness, for these results the proofs here contain laborious additions to the proofs in [FJMR], the present mathematical framework tells us more. Indeed we now allow the driving force in the Navier-Stokes equation to depend on time (although with some restrictions on the time fluctuations). The fact that the extension to the nonautonomous case is possible underscores the significance of using finite time averages, for it is clear that in many such situations, the invariant measures would not exist.

Another way in which this work differs from [FJMR] is the emphasis we place here on the averaged shell energy function \( e_{\kappa,2\kappa} \), which following [C], we call the Paley-Littlewood energy spectrum. The advantage in considering the Paley-Littlewood spectrum instead of \( S(\kappa) \) is that the former has a precise mathematical meaning in our framework, while the latter only a hypothetical one. Indeed the connection between the two is given by the following heuristic consideration: the averages \( e_{\kappa_1,\kappa_2} \) can be viewed as Riemann sums which “should” approximate the integral \( \int_{\kappa_1}^{\kappa_2} S(\kappa) d\kappa \) as the length of the spatial period goes to infinity [F97].

One remarkable fact about the Paley-Littlewood spectrum (proved in section 4) is that the global condition (1.2) connecting \( \kappa_\sigma \) and \( \kappa_\eta \) suffices to imply the following local property for \( \kappa \) in the inertial range

\[
(1.4) \quad e_{\kappa,2\kappa} \lesssim \frac{\eta^{2/3}}{\kappa^2} \ln \frac{\kappa_\eta}{\kappa}
\]

which is a slightly weaker version of one side of (1.1). Finally, in section 5, we show that deep in the dissipation range, the palinstrophy contained in the wave numbers larger than \( \kappa \) decays exponentially in \( \kappa \).

Although some previous works [BCFM,DF] have already pointed toward finite time averaged turbulence, this paper is the first systematic treatment along these lines. Finally, let us emphasize that while our results here as well as in [FJMR] are rigorous, they are largely guided by the heuristic arguments of classical turbulence.

1. Preliminaries

The incompressible Navier-Stokes equations

\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = F, \\
\text{div} u = 0, \\
\int \Omega u dx = 0, \\
u(x, 0) = u_0(x)
\]

with periodic boundary conditions in \( \Omega = [0, L]^2 \) can be written as a differential equation in a certain Hilbert space \( H \) (see [CF88] or [T97]),

\[
(1.1) \quad \frac{du}{dt} + \nu Au + B(u, u) = f(t), \quad u \in H.
\]
Here the space $H$ is the closure in $L^2(\Omega)^2$ of all $\mathbb{R}^2$-valued trigonometric polynomials $u$ such that

$$\nabla \cdot u = 0, \quad \text{and} \quad \int_\Omega u(x) dx = 0,$$

with scalar product

$$(u, v) = \int_\Omega u(x) \cdot v(x) dx, \quad \text{where} \quad a \cdot b = a_1 b_1 + a_2 b_2,$$

and associated norm

$$|u| = (u, u)^{1/2} = \left( \int_\Omega u(x) \cdot u(x) dx \right)^{1/2}.$$

The operator $A = -\Delta$ is self-adjoint with eigenvalues of the form

$$\left( \frac{2\pi}{L} \right)^2 k \cdot k, \quad \text{where} \quad k \in \mathbb{Z}^2 \setminus \{0\},$$

in increasing order, counted according to their multiplicities,

$$0 < \lambda_0 = \left( \frac{2\pi}{L} \right)^2 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

(Note $\lambda_j/\lambda_0 \in \mathbb{N}$.) The corresponding normalized eigenvectors are denoted $w_0, w_1, w_2, \ldots$, (that is $|w_j| = 1$ for $j = 1, 2, \ldots$). The bilinear operator $B$ is given by

$$B(u, v) = \mathcal{P}((u \cdot \nabla)v),$$

where $\mathcal{P}$ is the Helmholtz-Leray orthogonal projection of $L^2(\Omega)^2$ onto $H$.

We allow for a general time dependent force $f$, assuming at first, only that

$$\overline{|f|} = \sup_{t \in \mathbb{R}} |f(t)| < \infty.$$

More restrictions on $f$ will be imposed below and in Section 3.

The positive square root of $A$ is defined by

$$A^{1/2} w_j = \lambda_j^{1/2} w_j, \quad \text{for} \quad j = 0, 1, 2, \ldots$$

onto the set

$$V = \mathcal{D}_{A^{1/2}} = \{ u \in H : \sum_{j=0}^{\infty} \lambda_j (u, w_j)^2 < \infty \}. $$
The natural norm on $V$ will be
\[
\|u\| = |A^{1/2}u| = \left( \int_{\Omega} \frac{\partial}{\partial x_j} u(x) \cdot \frac{\partial}{\partial x_j} u(x) \, dx \right)^{1/2} = \left( \sum_{j=0}^{\infty} \lambda_j(u, w_j)^2 \right)^{1/2}.
\]

Observing the periodic boundary conditions, we may express the solution as a Fourier series
\[
(1.3) \quad u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{i \kappa_0 k \cdot x},
\]
where
\[
(1.4) \quad \kappa_0 = \lambda_0^{1/2}, \quad \hat{u}_0 = 0, \quad \hat{u}_k^* = \hat{u}_{-k},
\]
and due to incompressibility, $k \cdot \hat{u}_k = 0$. We associate to each term in (1.3) a wave number $\kappa_0 |k|$. Parseval’s identity reads as
\[
|u|^2 = L^2 \sum_{k \in \mathbb{Z}^2} \hat{u}_k \cdot \hat{u}_{-k} = L^2 \sum_{k \in \mathbb{Z}^2} |\hat{u}_k|^2,
\]
(we have also used $| \cdot |$ for the modulus of a vector in $\mathbb{C}^2$) or more generally as
\[
(u, v) = L^2 \sum_{k \in \mathbb{Z}^2} \hat{u}_k \cdot \hat{v}_{-k}.
\]
We define projectors $P_\kappa : H \to \text{span}\{w_j | \lambda_j \leq \kappa^2\}$ by
\[
P_\kappa u = \sum_{\kappa_0 |k| \leq \kappa} \hat{u}_k e^{i \kappa_0 k \cdot x},
\]
and set $Q_\kappa = I - P_\kappa$.

We will use the well-known orthogonality relation of the bilinear term
\[
(1.5) \quad (B(u, v), w) = -(B(u, w), v),
\]
which together with the strong enstrophy invariance
\[
(1.6) \quad (B(u, u), Av) = (B(Au, u), v)
\]
implies
\[
(1.7) \quad (B(u, u), Au) = 0.
\]
Recall Agmon’s inequality

\[ \|u\|_{\infty} \leq c_1 |u|^{1/2} |Au|^{1/2}, \]

and its alternative

\[ \|u\|_{\infty} \leq c_2 \left( \ln \frac{|Au|}{\kappa_0 \|u\|} + 1 \right)^{1/2} \|u\|, \]

(see [CF88, T97, DG]). The relations (1.5)-(1.9) are valid for any functions \( u, v, w \) for which all the operations are meaningful. The constants \( c_1, c_2 \), as well as lower case \( c_j, j = 3, 4, \ldots \) to follow, are absolute constants of the order of unity. Finally, we will need the following estimate which was proved in [FJMR], Lemma 1.1: for \( u, v, \) and \( w \) in \( V \), with \( P_\kappa w = w \)

\[ |(B(u, v), w)| \leq c_3 \left( \frac{1}{\kappa} + 1 \right)^{1/2} \|u\| \|v\| \|w\|, \]

where \( c_3 = \max\{2c_2, 12c_1\} \).

If we multiply (1.1) by \( u \), (respectively \( Au \)), integrate over \( \Omega \), and apply the orthogonality relations (1.5), (1.7) we find that

\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \nu \|u\|^2 = (f, u), \]

(1.11)

\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu |Au|^2 = (f, Au). \]

Straightforward applications of the Cauchy-Schwarz and Young inequalities to (1.12) gives

\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu |Au|^2 \leq \frac{\nu}{2} |Au|^2 + \frac{1}{2\nu} |f|^2, \]

(1.13)

The standard Gronwall inequality for (1.13) reads as

\[ \|u(t)\|^2 \leq e^{-\nu \kappa_0^2 (t-t_0)} \|u(t_0)\|^2 + \frac{|f|^2}{\nu^2 \kappa_0^2} \left( 1 - e^{-\nu \kappa_0^2 (t-t_0)} \right). \]

Thus the ball \( B \) of radius \( 2\nu \kappa_0 G \) in \( V \) where

\[ G = \frac{|f|}{\nu^2 \lambda_0} \]

is absorbing in that

\[ \|S(t_0 + t, t_0)u_0\| \leq 2\nu \kappa_0 G, \quad \text{for all } t \geq T(\|u_0\|). \]

(1.16)
Note also that \( \mathcal{B} \) is positively invariant, that is
\[
\|S(t_0 + t, t_0)u_0\| \leq 2\nu \kappa_0 G , \quad \text{for all } t > 0 , u_0 \in \mathcal{B}
\]
The adimensional \( G \) is the natural extension for a time dependent force of the generalized
Grashof number introduced in [FMTT]. We denote by \( S \) the solution operator defined
by \( S(t, t_0)u_0 = u(t; t_0, u_0) \), where \( u(t; t_0, u_0) \) is the unique solution to (1.1) such that
\( u(t_0; t_0) = u_0 \). We will often denote the solution as simply \( u(t) \), even in the nonautonomous
case, when the choice of \( t_0 \) and \( u_0 \) are understood. It is well known that under the
assumptions to this point, the operator
\[
S(t, t_0) : \mathcal{B}(\subset V) \mapsto V
\]
is compact for all \( t > t_0 \).

To compare to the ensemble average results in [FJMR], we will at times consider the
autonomous case, where \( f \) is independent of time. By setting \( S(t) = S(t + t_0, t_0) \) for all
\( t_0, t \) the global attractor \( \mathcal{A} \) can be defined by
\[
\mathcal{A} = \bigcap_{t \geq 0} S(t)\mathcal{B} .
\]
Equivalently \( \mathcal{A} \) is the largest bounded, invariant set (i.e. \( S(t)\mathcal{A} = \mathcal{A} \) for all \( t \geq 0 \)).

We assume that the force has the form
\[
f(t) = \sum_{\kappa \leq \kappa_0 |k| \leq \kappa} f_k(t)e^{i\kappa_0 k \cdot x} .
\]
where \( \kappa, \kappa \) are respectively, the largest and smallest wave numbers such that (1.18) holds.
The compactness of the operator
\[
S(t, t_0) : \mathcal{B}(\subset V) \mapsto \mathcal{D}_A
\]
is shown in [FT] to follow from condition (1.18).

We assume the following bound on the forcing range
\[
\kappa \leq C_0 \kappa_0 .
\]
Henceforth all constants depending only on \( C_0 \) (and other absolute constants) will be
denoted by uppercase \( C_j, j = 1, 2, \ldots \).

Let
\[
p_\kappa = P_\kappa u , \quad \text{and} \quad q_\kappa = Q_\kappa u ,
\]
and define
\[
\mathcal{E}_\kappa^\rightarrow(u) = -\frac{1}{L^2}(B(p_\kappa, p_\kappa), Aq_\kappa) \quad \text{and} \quad \mathcal{E}_\kappa^\leftarrow(u) = -\frac{1}{L^2}(B(q_\kappa, q_\kappa), Ap_\kappa)
\]
as the rates of enstrophy transfer or enstrophy fluxes from low to high, and high to low
wave numbers, respectively, at wave number \( \kappa \). Then \( \mathcal{E}_\kappa = \mathcal{E}_\kappa^\rightarrow - \mathcal{E}_\kappa^\leftarrow \) is the net rate of
enstrophy transfer (or net enstrophy flux) at the wave number \( \kappa \). It is shown in [FJMR]
that
\[
\frac{1}{2} \frac{d}{dt} \|q_\kappa\|^2 + \nu |Aq_\kappa|^2 = L^2 \mathcal{E}_\kappa + (f, Aq_\kappa) .
\]
2. Ensemble averages

The unpredictability of instantaneous quantities in fully developed turbulent flows dictates that they be averaged. Strictly speaking, for some continuous functions $\Phi$ and some initial data $u_0$, the infinite time limit, taken in the usual sense,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)u_0) d\tau,
$$

may not exist. By the Hahn-Banach theorem (see [DS1]), however, there exists a generalized limit denoted $\text{L} \text{im}_{t \to \infty}$, defined as a linear functional on

$$
B([0, \infty)) = \{g : g\text{ is a bounded real-valued function on } [0, \infty]\}
$$

which satisfies

$$
\lim_{t \to \infty} g(t) = \lim_{t \to \infty} g(t),
$$

whenever the right-hand side exists. Moreover, by the following result based on the Bogolyubov-Krylov theory ([KB]), and proved in [BCFM], we can consider the time limit as an ensemble average.

**2.1 Proposition.** For every $u_0 \in \mathcal{D}_A$ there exists an invariant probability measure $\mu_{u_0}$ such that

$$
\text{L} \text{im}_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)u_0) d\tau = \int_{\mathcal{A}} \Phi(u) \mu_{u_0}(du)
$$

for all real-valued continuous (with respect to the $H$-norm) functions $\Phi$ on $\mathcal{D}_A$.

The following result from [FJMR] says that any estimate which is valid for all measures $\mu_{u_0}$ is also valid for an arbitrary invariant probability measure.

**2.2 Lemma.** For any invariant probability measure $\mu$ on $\mathcal{D}_A$

$$
\int_{\mathcal{A}} \left[ \int_{\mathcal{A}} \Phi(u) \mu_{u_0}(du) \right] \mu(du_0) = \int_{\mathcal{A}} \Phi(u_0) \mu(du_0).
$$

As an aside let us mention that these invariant probability measures are the only stationary statistical solutions (in the sense of [F73ii]) of equation (1.1). In view of this last result, throughout the remainder of this section, we assume that an invariant measure $\mu$ is chosen, and that the averages, with respect to $\mu$ will be denoted as $\langle \cdot \rangle$, that is

$$
\langle \Phi(u) \rangle = \int_{\mathcal{A}} \Phi(u) \mu(du).
$$

Define the average dissipation rate of enstrophy (resp. energy) per unit mass by

$$
\eta = \frac{\nu}{L^2} \langle |Au|^2 \rangle, \quad \epsilon = \frac{\nu}{L^2} \langle \|u\|^2 \rangle.
$$
and wave numbers

\[
\kappa_\sigma = \left( \frac{\eta}{\epsilon} \right)^{1/2}, \quad \kappa_\eta = \left( \frac{\eta}{\nu^3} \right)^{1/6}, \quad \text{and} \quad \kappa_\epsilon = \left( \frac{\epsilon}{\nu^3} \right)^{1/4}.
\]

Heuristic arguments by Kolmogorov, Batchelor and Kraichnan associate the wave number \( \kappa_\eta \) with the dissipation range, where the transfer of enstrophy is dominated by viscous effects. A basic assumption in the Kraichnan theory is that the dissipation range starts at wave numbers that are much larger than those in the forcing. The heuristics arguments together with relations (1.20) then suggest that for the Kraichnan theory of fully developed turbulence to hold (as described by ensemble averages), we must have

\[
\kappa_\eta \gg \bar{\kappa}
\]

According to the rigorous estimate in [FMT93]

\[
\frac{1}{c_4} G^{1/6} \leq \frac{\kappa_\eta}{\kappa_0} \leq G^{1/3},
\]

we have that (2.5) is equivalent to

\[
G \gg 1.
\]

The condition (2.7) is independent of the solution, and thus can be readily verified in most applications, and easily realized in simulations. It will be an assumption in rigorous results to follow. Another main component of the Kraichnan theory is that within the so-called inertial range, which lies between the forcing and dissipation ranges, the enstrophy flux is nearly constant. Rigorous support for this cascade of enstrophy is given in [FJMR] with the estimate

\[
1 - \left( \frac{\kappa}{\kappa_\sigma} \right)^2 \leq \frac{\langle \mathcal{E}_\kappa \rangle}{\eta} \leq 1.
\]

It follows that if

\[
\kappa_\sigma \gg \bar{\kappa},
\]

we have the cascade relation

\[
\langle \mathcal{E}_\kappa \rangle \approx \eta, \quad \text{for} \quad \bar{\kappa} \leq \kappa \ll \kappa_\sigma.
\]
2.3 Remark. Kraichnan [K72] proposed a specific mechanism for (2.10) where eddies in the inertial range break up into eddies of about half their linear size while traveling a distance comparable to their linear size. This is mathematically stated as

\[
(2.11) \quad \langle \mathcal{E}_{\kappa} \rangle \approx \frac{1}{L^2} \langle -(B(u_{\kappa/2}, u_{\kappa/2}), A u_{\kappa, 2\kappa}) \rangle.
\]

A tentative outline of a proof of (2.11) presented in [FJMR] reads as

\[
(2.12) \quad \langle \mathcal{E}_{\kappa} \rangle \approx \langle \mathcal{E}_{\kappa} \rangle^* = \frac{1}{L^2} \langle -(B(u_{0, \kappa}, u_{0, \kappa}), A u_{\kappa, 2\kappa}) \rangle \\
\approx \frac{1}{L^2} \langle -(B(u_{\kappa/2}, u_{\kappa/2}), A u_{\kappa, 2\kappa}) \rangle.
\]

While the second relation in (2.12) is immediate, the first is proved in [FJMR] only deep in the dissipation range, and the third remains completely open.

While it is not known if the condition (2.9) is achievable, some relations between \( \kappa_\eta \) and \( \kappa_\sigma \) are established in [FJMR]. First it is proved that if (2.7) holds, then there exists a constant \( C_1 = C_1(\pi/\kappa_0) \) such that

\[
(2.13) \quad \kappa_\sigma \leq C_1 \kappa_\eta.
\]

In [FJMR] it is also shown that if Kraichnan’s theory of fully developed turbulence holds, then

\[
\kappa_\sigma \sim \kappa_\eta \left( \frac{\ln \frac{\kappa_\eta}{\kappa_i}}{\kappa_\eta} \right)^{-1/2},
\]

where \( \kappa_i \) is the lower end of the inertial range.

The well-known Kolmogorov relation in three-dimensional turbulence is

\[
(2.14) \quad \epsilon \sim \frac{U^3}{L}
\]

where

\[
U = \left( \frac{\langle |u|^2 \rangle}{L^2} \right)^{1/2}.
\]

The two-dimensional counterpart to (2.14) is

\[
\eta \sim \frac{U^3}{L^3}.
\]

One side of this relation, namely

\[
(2.15) \quad \eta \lesssim \frac{U^3}{L^3},
\]

is rigorously established in [FJMR]. For previous rigorous analogues of (2.15) see [CD94,CD95,F97,DF]

Our next task is to show that these rigorous results remain valid when the ensemble averages (2.4) are replaced by finite time averages along any solution in the absorbing ball \( \mathcal{B} \) (see (1.14)-(1.17)) with averaging time independent of the solution.
3. Finite time averages

Now consider for fixed \( u_0, t_0, t_1, \) and \( t_2, \) the finite time average

\[
\langle \Phi(u) \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \Phi(u(\tau)) \, d\tau.
\]

Henceforth let the average dissipation rates of enstrophy and energy per unit mass be

\[
\tilde{\eta} = \frac{\nu}{L^2} \langle |Au|^2 \rangle, \quad \tilde{\epsilon} = \frac{\nu}{L^2} \langle \|u\|^2 \rangle,
\]

with partial dissipation rates denoted

\[
\tilde{\eta}_{\kappa, \kappa'} = \frac{\nu}{L^2} \langle |Au_{\kappa, \kappa'}|^2 \rangle, \quad \tilde{\epsilon}_{\kappa, \kappa'} = \frac{\nu}{L^2} \langle \|u_{\kappa, \kappa'}\|^2 \rangle,
\]

and average energy (\( \times 2 \)) per unit mass in the eddies of length \( \ell \in [(2\kappa)^{-1}, \kappa^{-1}] \) given by

\[
\tilde{\epsilon}_{\kappa, 2\kappa} = \frac{1}{L^2} \langle |u_{\kappa, 2\kappa}|^2 \rangle.
\]

To ease the exposition we mention that similar expressions, without the factor of 2, and with \( | \cdot | \) replaced respectively by \( \| \cdot \| \) and \( |A \cdot | \), define the enstrophy and palinstrophy, per unit mass, in the eddies over the same length scales. Since we will deal only with the dissipation rate of enstrophy, rather than with enstrophy or palinstrophy themselves, we will not introduce additional notation for the latter two quantities.

As in the case of ensemble averages, we also define wave numbers

\[
\tilde{\kappa}_{\sigma} = \left( \frac{\tilde{\eta}}{\tilde{\epsilon}} \right) \frac{1}{2}, \quad \tilde{\kappa}_{\eta} = \left( \frac{\tilde{\eta}}{\nu^3} \right)^{1/6}, \quad \tilde{\kappa}_{\epsilon} = \left( \frac{\tilde{\epsilon}}{\nu^3} \right)^{1/4}, \quad \text{and} \quad \tilde{\kappa}_{\tau} = \left( \frac{\langle \|u\|^2 \rangle}{\langle \|u\|^2 \rangle} \right)^{1/2},
\]

the last being the Taylor wave number, primarily used in the study of three-dimensional turbulence.

We begin with finite time version of \((2.6)\).

3.1 Proposition. Suppose that for some \( \Gamma_1, \Gamma_2 > 0 \)

\[
\langle |\hat{f}|^2 \rangle \leq \Gamma_1 \nu^2 \kappa_0^4 \langle |f|^2 \rangle, \quad \Gamma_2 |f|^2 \leq \langle |f|^2 \rangle,
\]

where \( \hat{f} = df/dt, \)

\[
c_3 G \geq 1 + \frac{\Gamma_1}{2},
\]

and that

\[
t_2 - t_1 \geq T_2 = \frac{8L^2}{\nu \Gamma_2 \pi^2}.
\]
Then

\[(3.5) \quad \left( \frac{G}{C_2} \right)^{1/6} \leq \frac{\tilde{\kappa}_1}{\kappa_0} \leq G^{1/3}, \]

where

\[C_2 = 32\pi^2 \frac{c_3}{\Gamma_2} \left( \left[ \ln \frac{\kappa}{\kappa_0} + 1 \right]^{1/2} + 1 \right).\]

(If \( \tilde{f} = 0 \), one takes \( \Gamma_1 = 0, \Gamma_2 = 1 \).)

Proof.

Integrate (1.12) to obtain

\[(3.6) \quad \frac{\nu \left[ \|u(t_2)\|^2 - \|u(t_1)\|^2 \right]}{L^2(t_2 - t_1)} + \frac{\nu \tilde{\eta}}{L^2} = \frac{\nu \langle (Au, f) \rangle^{\tilde{\eta}}}{L^2}.\]

Apply the Cauchy-Schwarz and Jensen’s inequalities along with (1.17) to make the estimate

\[\nu \tilde{\eta} \leq \frac{\nu \|u(t_1)\|^2}{L^2(t_2 - t_1)} + \frac{\nu^{1/2} \|f\|}{L} \left\{ \frac{\nu}{L^2} \langle |Au|^2 \rangle \right\}^{1/2} \]

\[\leq \frac{4\nu^3 \kappa_0^2 G^2}{L^2(t_2 - t_1)} + \frac{\nu^{1/2} \|f\|}{L} \tilde{\eta}^{1/2}.\]

By (1.2), (1.4) this can be rewritten as

\[\tilde{\eta} \leq \frac{16\pi^2 \nu^3 G^2}{L^4(t_2 - t_1)} + \frac{4\pi^3 \nu^3 G^2}{L^3} \tilde{\eta}^{1/2}.\]

Use the quadratic formula to obtain

\[(3.7) \quad \tilde{\eta}^{1/2} \leq \frac{1}{2} \left\{ \frac{4\pi^2 \nu^{3/2} G}{L^3} + \left[ \frac{16\pi^4 \nu^3 G^2}{L^6} + \frac{64\pi^2 \nu^2 G^2}{L^4(t_2 - t_1)} \right]^{1/2} \right\}.

It follows from (3.2) that \( \Gamma_2 \leq 1 \) so that

\[\frac{8L^2}{\nu \Gamma_2 \pi^2} \geq \frac{64L^2}{9\nu \pi^2},\]

and consequently by (3.4), we have

\[(3.8) \quad \frac{1}{t_2 - t_1} \leq \frac{9\nu \pi^2}{64L^2}.\]
Inserting (3.8) into (3.7) gives
\[ \tilde{\eta}^{1/2} \leq \frac{9 \pi^2 \nu^{3/2} G}{2 L^3}. \]
Multiply by \( L^3 (2\pi)^{-3} \nu^{-3/2} \), take the cube root, and apply (1.4) to reach
\[ \frac{\tilde{\kappa} \eta}{\kappa_0} \leq \left( \frac{9 G}{16\pi} \right)^{1/3}, \]
which completes the proof of the upper bound in (3.5).

The scalar product of (1.1) with \( f \) gives
\[ \frac{d}{dt} (u, f) + \nu (Au, f) + (B(u, u), f) = |f|^2 + (u, \dot{f}), \]
from which follows
\[ \frac{(u, f)(t_2) - (u, f)(t_1)}{L^2(t_2 - t_1)} + \frac{\nu}{L^2} \langle (Au, f) \rangle + \frac{1}{L^2} \langle (B(u, u), f) \rangle = \frac{\langle |f|^2 \rangle}{L^2} + \frac{1}{L^2} \langle (u, \dot{f}) \rangle. \]
Add (3.6) and (3.9) to find
\[ \delta_1 + \delta_2 + \nu \tilde{\eta} + \frac{1}{L^2} \langle (B(u, u), f) \rangle = \frac{\langle |f|^2 \rangle}{L^2} + \frac{1}{L^2} \langle (u, \dot{f}) \rangle, \]
where
\[ \delta_1 = \frac{\nu \left[ \| u(t_2) \|^2 - \| u(t_1) \|^2 \right]}{L^2(t_2 - t_1)} \quad \text{and} \quad \delta_2 = \frac{(u, f)(t_2) - (u, f)(t_1)}{L^2(t_2 - t_1)}. \]

Apply the Cauchy-Schwarz inequality, first in space, then in time, followed by (3.3), then Young's inequality to get
\[ \frac{1}{L^2} |\langle (u, \dot{f}) \rangle| \leq \frac{\Gamma_1 \nu^2 \kappa_0^4}{2 L^2} \langle |u|^2 \rangle + \frac{\langle |f|^2 \rangle}{2 L^2} \]
\[ \leq \frac{\Gamma_1 \nu^2 \kappa_0^4}{2 L^2} \frac{1}{2} \nu \tilde{\eta} + \frac{\langle |f|^2 \rangle}{2 L^2}. \]
Use this and (1.10) in (3.10) to obtain
\[ \frac{\langle |f|^2 \rangle}{2 L^2} \leq \delta_1 + \delta_2 + \nu \left( 1 + \frac{\Gamma_2}{2} \right) \tilde{\eta} + \frac{c_3}{\nu} \left[ \ln \frac{\kappa}{\kappa_0} + 1 \right]^{1/2} \tilde{\eta}. \]
Use the Cauchy-Schwarz and Poincaré inequalities, to make the estimate
\[ \frac{\Gamma_2}{2} \frac{|f|^2}{L^2} \leq \frac{\langle |f|^2 \rangle}{2 L^2} \leq \frac{\nu \| u(t_2) \|^2}{L^2(t_2 - t_1)} + \frac{2 \| u \| |f|}{\kappa_0 L^2(t_2 - t_1)} + \frac{c_3}{\nu} \left[ \ln \frac{\kappa}{\kappa_0} + 1 \right]^{1/2} \frac{|f|^2}{\kappa_0^2} \tilde{\eta} + \left( 1 + \frac{\Gamma_1}{2} \right) \nu \tilde{\eta}. \]
and then (1.17), (1.15), (1.2), (1.4) to arrive to
\[
\frac{8\pi^4 \nu^4 \Gamma_2}{L^6} G^2 \leq \frac{32\pi^2 \nu^3}{L^4 (t_2 - t_1)} G^2 + \left( c_3 \left[ \ln \frac{\bar{\kappa}}{\kappa_0} + 1 \right]^{1/2} \right) \nu \hat{\eta} + \left( 1 + \frac{\Gamma_1}{2} \right) \nu \hat{\eta} .
\]
The condition (3.4) implies
\[
\frac{32\pi^2 \nu^3}{L^4 (t_2 - t_1)} \leq \frac{4\pi^4 \nu^4 \Gamma_2}{L^6}
\]
so that by (3.3)
\[
\frac{4\pi^4 \nu^4 \Gamma_2}{L^6} G^2 \leq \left( c_3 \left[ \ln \frac{\bar{\kappa}}{\kappa_0} + 1 \right]^{1/2} G + 1 + \frac{\Gamma_1}{2} \right) \nu \hat{\eta} \leq 2c_3 \left( \left[ \ln \frac{\bar{\kappa}}{\kappa_0} + 1 \right]^{1/2} + 1 \right) G \nu \hat{\eta} ,
\]
and hence
\[
\frac{2\pi^4 \nu^3 \Gamma_2 G}{L^6 c_3 \left( \left[ \ln \frac{\bar{\kappa}}{\kappa_0} + 1 \right]^{1/2} + 1 \right)} \leq \hat{\eta} .
\]
Use (1.4) to rewrite the last estimate as
\[
\frac{G}{C_2} \leq \frac{\hat{\eta}}{\kappa_0^6 \nu^3} ,
\]
which gives the the lower bound in (3.5).

We can now prove the analogue of the cascade estimate (2.8) in the case of finite time averages.

**3.2 Proposition.** Assume that (3.2), (3.3), and (3.4) hold. Then for any \( \varepsilon > 0 \) and \( \kappa > \bar{\kappa} \) we have
\[
1 - \frac{\kappa^2}{\bar{\kappa}^2} - \varepsilon \leq \frac{\langle \mathcal{E}_\kappa \rangle}{\hat{\eta}} \leq 1 + \varepsilon ,
\]
provided that
\[
t_2 - t_1 \geq \frac{C_2 L^2 G}{8\pi^4 \nu \varepsilon} .
\]
Thus the enstrophy cascade holds for finite time averages, provided the time interval is long enough.

**Proof.**
Integrate (1.22), and observe the condition $\kappa \geq \kappa$ to obtain

$$\langle \mathcal{E}_\kappa \rangle - \hat{\eta}_{\kappa, \infty} = \frac{1}{2L^2(t_2 - t_1)} \left( \|q_\kappa(t_2)\|^2 - \|q_\kappa(t_1)\|^2 \right)$$

Use (1.17) to reach

$$\langle \mathcal{E}_\kappa \rangle - \hat{\eta}_{\kappa, \infty} \leq \frac{\|q_\kappa(t_2)\|^2}{2L^2(t_2 - t_1)} \leq \frac{8\pi^2 \nu^2 G^2}{L^4(t_2 - t_1)},$$

and $|Ap_\kappa|^2 \leq \kappa^2 \|p_\kappa\|^2 \leq \kappa^2 \|u\|^2$ to write

$$\hat{\eta} \geq \hat{\eta}_{\kappa, \infty} = \hat{\eta} - \frac{\nu}{L^2} \langle |Ap_\kappa|^2 \rangle$$

$$\geq \hat{\eta} - \kappa^2 \hat{\epsilon}$$

$$= \hat{\eta} \left[ 1 - \frac{\kappa^2}{\kappa^2} \right].$$

It follows from (3.15) and (3.16) that

$$\hat{\eta} \left[ 1 - \frac{\kappa^2}{\kappa^2} \right] - \frac{8\pi^2 \nu^2 G^2}{L^4(t_2 - t_1)} \leq \langle \mathcal{E}_\kappa \rangle \leq \hat{\eta} + \frac{8\pi^2 \nu^2 G^2}{L^4(t_2 - t_1)}.$$ 

The lower bound in Proposition 3.1 is equivalent to

$$\frac{1}{\hat{\eta}} \leq \frac{C_2}{\kappa_0^2 G^3 \nu^3},$$

which can be combined with (3.17) in order to obtain

$$1 - \frac{\kappa^2}{\kappa^2} - \frac{C_2 L^2 G}{8\pi^4 \nu(t_2 - t_1)} \frac{\langle \mathcal{E}_\kappa \rangle}{\hat{\eta}} \leq 1 + \frac{C_2 L^2 G}{8\pi^4 \nu(t_2 - t_1)}.$$ 

The next estimate relates $\hat{\kappa}_{\sigma}$ to $\hat{\kappa}_{\eta}$ along the lines of (2.13), and quantifies the perturbation due to averaging over a finite time interval.

3.3 Theorem. Assume that (2.7), (3.2), (3.3), and (3.4) hold, along with (3.14) for

$$\varepsilon \leq \left[ 4\kappa_0^6 \nu^3 \left( 1 + \frac{2}{\Gamma_2} C_0 \right) \right]^{-1}.$$ 

Then

$$\hat{\kappa}_{\sigma}^3 \left( 1 - \frac{\delta}{\hat{\eta}} \right) \leq C_3 \hat{\kappa}_{\eta}^3.$$
where
\[ C_3 = \frac{8\pi c_3}{\Gamma_2} \frac{\pi}{\kappa_0} \left[ \ln \frac{\pi}{\kappa_0} + 1 \right]^{1/2}, \]

and
\[
\frac{\delta}{\bar{\nu}} \leq \frac{C_2 \left( 1 + \frac{2}{\Gamma_2} C_0 \right) G}{\pi^2 \nu \kappa_0^2 (t_2 - t_1)}.
\]

**Proof.** Let \( \delta_1 \) and \( \delta_2 \) be as in (3.11). Rearrange (3.6), then apply the Cauchy-Schwarz and Jensen inequalities to make the estimate
\[
\nu \tilde{\eta} = -\delta_1 + \frac{\nu}{L^2} \langle (Au, f) \rangle \leq -\delta_1 + \bar{\kappa} \nu^{1/2} \bar{\epsilon}^{1/2} \frac{|f|}{L}.
\]

Use (3.22) in (3.12) to reach
\[
\frac{\Gamma_2 |f|^2}{2L^2} \leq \frac{\langle |f|^2 \rangle}{2L^2} \leq \delta_2 + \bar{\kappa} \nu^{1/2} \bar{\epsilon}^{1/2} \left( 1 + \frac{\Gamma_2}{2} \right) \frac{|f|}{L} + \frac{c_3}{\nu} \left[ \ln \frac{\pi}{\kappa_0} + 1 \right]^{1/2} \bar{\epsilon} |f|.
\]

Now use (3.23) in (3.22) to obtain
\[
\nu \tilde{\eta} \leq -\delta_1 + \frac{2\bar{\kappa} \nu^{1/2} \bar{\epsilon}^{1/2}}{\Gamma_2} \left\{ \frac{\delta_2 L}{|f|} + \left( 1 + \frac{\Gamma_2}{2} \right) \bar{\kappa} \nu^{1/2} \bar{\epsilon}^{1/2} + \frac{c_3 L}{\nu} \left[ \ln \frac{\pi}{\kappa_0} + 1 \right]^{1/2} \bar{\epsilon} \right\},
\]

which can be rewritten as
\[
\tilde{\eta} \leq \delta + \frac{2\bar{\kappa}^2}{\Gamma_2} \left( 1 + \frac{\Gamma_2}{2} \right) \bar{\epsilon} + \frac{4\pi c_3}{\Gamma_2} \frac{\bar{\kappa}}{\kappa_0} \left( \frac{\bar{\epsilon}}{\nu} \right)^{3/2} \left[ \ln \frac{\pi}{\kappa_0} + 1 \right]^{1/2},
\]

where
\[
\delta = -\frac{\delta_1}{\nu} + 2\bar{\kappa} \frac{\delta_2}{\Gamma_2} \left( \frac{\bar{\epsilon}}{\nu} \right)^{1/2} \frac{L}{|f|}.
\]

Suppose
\[
\frac{2\bar{\kappa}^2}{\Gamma_2} \left( 1 + \frac{\Gamma_2}{2} \right) \bar{\epsilon} > \frac{4\pi c_3}{\Gamma_2} \frac{\bar{\kappa}}{\kappa_0} \left( \frac{\bar{\epsilon}}{\nu} \right)^{3/2} \left[ \ln \frac{\pi}{\kappa_0} + 1 \right]^{1/2}.
\]

Then, since \( \Gamma_2 \leq 1 \) and \( c_1 \geq 1 \), we have
\[
\bar{\kappa} \kappa_0 > \left( \frac{\bar{\epsilon}}{\nu^3} \right)^{1/2}.
\]
Yet, if (3.26) holds, then by (3.24) and (3.27)
\[ \tilde{\eta} \leq \delta + \frac{4\pi^4 \kappa_0^2 \nu^3}{\Gamma_2} \left( 1 + \frac{\Gamma_2}{2} \right), \]
which by the lower bound in Proposition 3.1, implies
\[ G \leq \frac{\delta C_2}{\kappa_0^6 \nu^3} + \left( \frac{\kappa_0}{\kappa} \right)^4 \frac{4C_2}{\Gamma_2} \left( 1 + \frac{\Gamma_2}{2} \right). \]
We show below that
\[ \delta \leq \frac{16\pi^2 \nu^2 G^2}{L^4(t_2 - t_1)} \left( 1 + \frac{2}{\Gamma_2 C_0} \right). \]
which by (3.19) gives
\[ \frac{1}{2} G \leq \left( \frac{\kappa}{\kappa_0} \right)^4 \frac{4C_2}{\Gamma_2} \left( 1 + \frac{\Gamma_2}{2} \right), \]
contradicting (2.7).

Hence, the reverse inequality must hold in (3.26), which when used in (3.24) implies
\[ \tilde{\eta} \leq \delta + 2 \frac{8\pi C_3 \kappa}{\Gamma_2} \left( \frac{\tilde{\epsilon}}{\nu} \right)^{3/2} \left[ \ln \frac{\kappa}{\kappa_0} + 1 \right]^{1/2}, \]
Multiply (3.30) by \( \tilde{\eta}/\tilde{\epsilon}^{3/2} \) to obtain
\[ \left( \frac{\tilde{\eta}}{\tilde{\epsilon}} \right)^{3/2} \leq \delta \tilde{\eta}^{1/2} \frac{3\nu \kappa}{\Gamma_2} \left( \frac{\tilde{\eta}}{\nu^3} \right)^{1/2} \left[ \ln \frac{\kappa}{\kappa_0} + 1 \right]^{1/2}, \]
which is easily converted to (3.20).

To estimate \( \delta \), use (1.17) to obtain
\[ \tilde{\epsilon} \leq \frac{4\nu^3 G^2 \kappa_0^2}{L^2}, \]
and then \( |(u, f)| \leq \|u\| \|f\| / \kappa \), (1.17) twice more, and finally (3.32), to make the estimates
\[ \delta \leq \frac{1}{L^2(t_2 - t_1)} \left[ \|u(t_1)\|^2 + \frac{2\pi \tilde{\epsilon}^{1/2} L}{\Gamma_2 \nu^{1/2}} \|u(t_2)\| + \|u(t_1)\| \right] \]
\[ \leq \frac{G}{L^2(t_2 - t_1)} \left[ 2\nu \kappa_0 \|u(t_1)\| + \frac{4}{\Gamma_2} \tilde{\epsilon}^{1/2} L \kappa_0 \right] \]
\[ \leq \frac{G}{L^2(t_2 - t_1)} \left[ 4\nu^2 \kappa_0^2 G + \frac{4}{\Gamma_2} \tilde{\epsilon}^{1/2} L \kappa_0 \right] \]
\[ \leq \frac{4\nu^2 \kappa_0^2 G^2}{L^2(t_2 - t_1)} \left[ 1 + \frac{2}{\Gamma_2 \kappa_0} \right]. \]
Applying the lower bound in Proposition 3.1 gives (3.21).

\[ \square \]

We conclude this section with the analogue of (2.15) for finite time averages.
3.4 Theorem. If
\[ \frac{C_2(3C_0 + 1)G}{\nu \kappa_0^2 (2\pi)^2 (t_2 - t_1)} \leq \varepsilon < 1 , \]
we have
\[ \tilde{\eta} \leq \frac{C_4}{1 - \varepsilon} \tilde{U}^3 \]
where
\[ \tilde{U} = \left( \frac{\langle |u|^2 \rangle}{L^2} \right)^{1/2} \]

Proof. A straightforward adaptation of the proof of (2.15) in [FJMR] gives
\[ \tilde{\eta} \leq C_4 \frac{\tilde{U}^3}{L^3} + \frac{(3C_0 + 1)(2\pi)^2 \nu^2 G^2}{L^4 (t_2 - t_1)} . \]
By the lower bound in Proposition 3.1 we have
\[ \tilde{\eta} \geq \frac{G}{C_2} \nu^3 \kappa_0^6 . \]
and hence
\[ 1 \leq C_4 \frac{\tilde{U}^3}{\tilde{\eta} L^3} + \frac{C_2(3C_0 + 1)G}{\nu \kappa_0^2 (2\pi)^2 (t_2 - t_1)} \leq C_4 \frac{\tilde{U}^3}{\tilde{\eta} L^3} + \varepsilon . \]
Solve for \( \tilde{\eta} \) to obtain (3.33).

\[ \square \]

4. The energy spectrum

We establish in this section several relations between the energy spectrum and \( \tilde{\kappa}_\sigma, \tilde{\kappa}_\eta, \) and \( \tilde{\kappa}_r \).

**Lemma 4.1.** The following relations hold for all \( \kappa \leq \kappa' \)
\[ \frac{\tilde{\varepsilon}_{\kappa', \infty}}{\tilde{\varepsilon}_{\kappa, \kappa'}} \leq \frac{\tilde{\varepsilon}_{\kappa', \infty}}{\tilde{\varepsilon}_{\kappa, \kappa'}} \leq \frac{\tilde{\eta}_{\kappa', \infty}}{\tilde{\eta}_{\kappa, \kappa'}} . \]
Thus if
\[ \tilde{\eta}_{\kappa', \infty} \ll \tilde{\eta}_{\kappa, \infty} , \]
then
\[ \tilde{\varepsilon}_{\kappa', \infty} \ll \tilde{\varepsilon}_{\kappa, \infty} , \quad \tilde{\varepsilon}_{\kappa', \infty} \ll \tilde{\varepsilon}_{\kappa, \infty} . \]
(4.4) \[ \frac{\tilde{e}_{k', \infty}}{\nu \tilde{e}_{k, \infty}} \approx \frac{\tilde{e}_{k, k'}}{\nu \tilde{e}_{k', k'}} , \text{ and } \frac{\tilde{\eta}_{k, \infty}}{\tilde{e}_{k, \infty}} \approx \frac{\tilde{\eta}_{k', \infty}}{\tilde{e}_{k', \infty}} . \]

Proof. The second relation in (4.1) is proved by the following

\[ \frac{\tilde{e}_{k', \infty}}{\tilde{e}_{k, k'}} = \left( \frac{\|u_{k', \infty}\|}{\nu} \right)^2 \leq \frac{\|Au_{k', \infty}\|}{\nu^2 \|u_{k', \infty}\|^2} \leq \frac{\|Au_{k', \infty}\|}{\nu^2 \|u_{k', \infty}\|^2} = \frac{\tilde{\eta}_{k', \infty}}{\tilde{\eta}_{k', \infty}} . \]

Since \( \tilde{\eta}_{k, \infty} = \tilde{\eta}_{k, k'} + \tilde{\eta}_{k', \infty} \), the second relation in (4.1) combined with (4.2) gives

\[ \frac{\tilde{e}_{k', \infty}}{\tilde{e}_{k, k'}} \leq \frac{\tilde{\eta}_{k', \infty}}{\tilde{\eta}_{k', \infty}} \ll 1 , \]

which, combined with \( \tilde{e}_{k, \infty} = \tilde{e}_{k, k'} + \tilde{e}_{k', \infty} \), gives the second relation in (4.3). The second relation in (4.4) follows from

\[ \frac{\tilde{\eta}_{k, \infty}}{\tilde{e}_{k, \infty}} = \frac{\tilde{\eta}_{k, k'} + \tilde{\eta}_{k', \infty}}{\tilde{e}_{k, k'} + \tilde{e}_{k', \infty}} . \]

The relations involving \( \tilde{e} \) are proved in a similar fashion.

Lemma 4.2. If

(4.5) \[ \pi \leq 2k_i < \theta_1 \tilde{\kappa}_{\tau} \quad \text{and} \quad \pi \leq 2k_i < \theta_2 \tilde{\kappa}_{\sigma} \]

for some \( 0 < \theta_1, \theta_2 \ll 1 \), then

(4.6) \[ \frac{\tilde{\eta}_{2k_i, \infty}}{\tilde{e}_{2k_i, \infty}} \sim \frac{\tilde{\eta}}{\tilde{e}} . \]

Proof. Set \( \kappa = 2k_i \) in (3.16) to obtain

\[ \tilde{\eta} \geq \tilde{\eta}_{2k_i, \infty} \geq \left( 1 - \left( \frac{2k_i}{\tilde{\kappa}_{\sigma}} \right)^2 \right) \tilde{\eta} \geq (1 - \theta_2^2) \tilde{\eta} , \]

and hence

(4.7) \[ 1 - \theta_2^2 \leq \frac{\tilde{\eta}_{2k_i, \infty}}{\tilde{\eta}} \leq 1 . \]

A similar estimate to (3.16), but for \( \tilde{e} \) yields

\[ \tilde{e} \geq \tilde{e}_{2k_i, \infty} \geq \left( 1 - \left( \frac{2k_i}{\tilde{\kappa}_{\tau}} \right)^2 \right) \tilde{e} \geq (1 - \theta_1^2) \tilde{e} , \]
and consequently

\[
1 - \theta_1^2 \leq \frac{\tilde{c}_{2\kappa,\infty}}{\tilde{c}} \leq 1.
\]

Dividing the terms in (4.7) by those in (4.8) we find that

\[
1 - \theta_2^2 \leq \frac{\tilde{\eta}_{2\kappa,\infty}}{\tilde{c}_{2\kappa,\infty}} \frac{\tilde{c}}{\tilde{\eta}} \leq \frac{1}{1 - \theta_1^2},
\]

which completes the proof.

We can now give the main result in this section. In essence this result shows that if \( \tilde{\kappa}_\tau/\bar{\kappa}, \tilde{\kappa}_\sigma/\bar{\kappa}, \) and \( \tilde{\eta}/\tilde{\eta}_{2\kappa,\bar{\kappa}} \) are large enough, then in the study of the energy spectrum in the inertial range, the behavior of that spectrum within the inertial range can be neglected.

**Proposition 4.3.** If (4.5) holds along with

\[
\tilde{\eta}_{\tilde{\kappa},\infty} \ll \tilde{\eta}
\]

then

\[
\tilde{\eta} \sim \frac{\tilde{\eta}_{2\kappa,\tilde{\kappa}}}{{\tilde{c}_{2\kappa,\tilde{\kappa}}}}.
\]

**Proof.** By (4.7) we have \( \tilde{\eta}_{2\kappa,\infty} \sim \tilde{\eta} \) and hence by (4.9) we have \( \tilde{\eta}_{\tilde{\kappa},\infty} \ll \tilde{\eta}_{2\kappa,\infty} \). Apply Lemma 4.1 with \( \kappa' = \tilde{\kappa}, \kappa = 2\kappa, \) and then Lemma 4.2 to obtain

\[
\frac{\tilde{\eta}_{2\kappa,\infty}}{\tilde{c}_{2\kappa,\infty}} \sim \frac{\tilde{\eta}_{2\kappa,\tilde{\kappa}}}{\tilde{c}_{2\kappa,\tilde{\kappa}}} \sim \frac{\tilde{\eta}}{\tilde{c}}.
\]

We next assume that the shell spectrum with a logarithm correction holds, namely that

\[
\tilde{c}_{2\kappa,\kappa} \sim \frac{\tilde{\eta}_{2}^{2/3}}{\kappa^2} \left( \ln \frac{\kappa}{\kappa_i} \right)^{-\alpha}, \quad \text{for some } \alpha \in [0, 1) \text{ and all } 2\kappa_i < \kappa < \bar{\kappa}_i \approx \tilde{\kappa}_\eta.
\]

With \( \alpha = 0 \) this is the original 2-D energy spectrum of Batchelor and Kraichnan; \( \alpha = 1/3 \) was also considered by Kraichnan (see [O]).

**Proposition 4.4.** If (4.10) holds, then

\[
\tilde{c}_{2\kappa,\tilde{\kappa}_\eta} \sim \frac{\tilde{\eta}_{2}^{2/3}}{\kappa_i^2}.
\]
If, in addition,

\[(4.12) \quad \kappa_i \ll \hat{\kappa}_\eta, \]

then

\[(4.13) \quad \hat{\eta}_{2\kappa_i, \hat{\kappa}_\eta} \sim \hat{\eta} \left( \ln \left( \frac{\hat{\kappa}_\eta}{\kappa_i} \right) \right)^{-\alpha}, \quad \hat{\epsilon}_{2\kappa_i, \hat{\kappa}_\eta} \sim \nu \hat{\eta}^{2/3} \left( \ln \left( \frac{\hat{\kappa}_\eta}{\kappa_i} \right) \right)^{1-\alpha}. \]

Proof. Let \( n \) satisfy

\[(4.14) \quad 2^n \kappa_i \leq \hat{\kappa}_\eta \leq 2^{n+1} \kappa_i. \]

Sum over all \( \kappa = 2^j \kappa_i \) for \( j = 1, 2, \ldots, n \), and apply (4.10) to obtain

\[ \hat{\epsilon}_{2\kappa_i, \hat{\kappa}_\eta} \sim \sum \hat{\epsilon}_\kappa \sim \hat{\eta}^{2/3} \sum \kappa^{-2} \left( \ln \left( \frac{\kappa}{\kappa_i} \right) \right)^{-\alpha} \sim \hat{\eta}^{2/3} \sum \left( \frac{\kappa}{\kappa_i} \right)^2 \left( \ln \left( \frac{\kappa}{\kappa_i} \right) \right)^{-\alpha} \sim \hat{\eta}^{2/3} \frac{\kappa_i^2}{\kappa^2}. \]

Similarly sum each expression in

\[ \frac{1}{16L^2} \langle |A_{\kappa, 2\kappa}|^2 \rangle \leq \kappa^4 c_\kappa \leq \frac{1}{L^2} \langle |A_{\kappa, 2\kappa}|^2 \rangle \]

over all \( \kappa = 2^j \kappa_i \) for \( j = 1, 2, \ldots, n \), and apply (4.10) to obtain

\[ \frac{1}{L^2} \langle |A_{2\kappa_i, 2^n \kappa_i}|^2 \rangle \sim \sum \kappa^4 \hat{\epsilon}_\kappa \sim \sum \hat{\eta}^{2/3} \kappa^2 \left( \ln \left( \frac{\kappa}{\kappa_i} \right) \right)^{-\alpha} \]

\[ = \hat{\eta}^{2/3} \frac{\kappa_i^2}{\kappa^2} \sum \left( \frac{\kappa}{\kappa_i} \right)^2 \left( \ln \left( \frac{\kappa}{\kappa_i} \right) \right)^{-\alpha} = \hat{\eta}^{2/3} \frac{\kappa_i^2}{\kappa^2} S_n \]

where \( S_n = \sum_{j=1}^n 4^j \ln \left( 2^j \right)^{-\alpha}. \)

Note that with \( \beta = \ln 4 \) we have

\[ (\ln 2)^{-\alpha} \int_0^n e^{\beta x} x^{-\alpha} dx \leq S_n \leq (\ln 2)^{-\alpha} \int_1^{n+1} e^{\beta x} x^{-\alpha} dx. \]

Apply L’Hospital’s rule to find the limit

\[ \lim_{y \to \infty} \frac{\int_y^n e^{\beta y} x^{-\alpha} dx}{e^{\beta y} y^{-\alpha}} = \lim_{y \to \infty} \frac{1}{\beta - \alpha y^{-1}} = \frac{1}{\beta}, \]

and hence that

\[ (\ln 2)^{-\alpha} \int_1^n e^{\beta x} x^{-\alpha} dx \approx \frac{1}{\ln 4} 4^n (n \ln 2)^{-\alpha} = \frac{1}{\ln 4} 4^n \left( \ln \left( \frac{\hat{\kappa}_\eta}{\kappa_i} \right) \right)^{-\alpha}, \] for \( \kappa_i \ll \hat{\kappa}_\eta. \)
It follows that
\[
\frac{1}{L^2} \langle |Au_{2\kappa_i,2^n\kappa_i}|^2 \rangle \sim \hat{\eta}^{2/3} n_{\kappa_i}^2 \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-\alpha} \sim \hat{\eta}^{2/3} \kappa_i^2 \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-\alpha} = \frac{\hat{\eta}}{\nu} \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-\alpha},
\]
which gives the first relation in (4.13).

Similarly, we have
\[
\frac{1}{L^2} \langle \|u_{2\kappa_i,2^n\kappa_i}\|^2 \rangle \sim \sum \kappa_i^2 \hat{c}_\kappa \sim \hat{\eta}^{2/3} \sum_1^n j^{-\alpha} \sim n^{-\alpha} \sim \hat{\eta}^{2/3} \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{1-\alpha}
\]
which proves the second relation in (4.13).

**Proposition 4.5.** If (4.5), (4.9), and (4.12) hold, then
\[
\hat{\kappa}_\sigma \sim \hat{\kappa}_\eta \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-1/2}.
\]

*Proof.* Apply Propositions 4.3 and 4.4 to obtain
\[
\hat{\kappa}_\sigma = \frac{\hat{\kappa}_\sigma}{\hat{\kappa}_\eta} \sim \frac{\hat{\eta} \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-\alpha}}{\hat{\eta}^{2/3} \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-\alpha}} = \frac{\hat{\eta}^{1/3}}{\nu} \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-1} = \frac{\hat{\kappa}_\sigma}{\hat{\kappa}_\eta} \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-1}.
\]

\[
\hat{\eta} \lesssim \hat{c}_\kappa^{3/2} \kappa_\kappa^{3} \lesssim \hat{c}_\kappa^{3/2} \kappa_i^{3}
\]
For the next result we assume that both sides of the Kolmogorov relation hold, that is,
\[
\hat{\eta} \sim \hat{c}_\kappa^{3/2} \kappa_\kappa^{3}.
\]

**Theorem 4.6.** If the three basic assumptions in the Kraichnan theory hold, namely (4.12), (4.9), and (4.10), together with (4.5), then
\[
\frac{\hat{\kappa}_\sigma}{\kappa_i} \sim \frac{\hat{\kappa}_\eta}{\kappa_i} \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{-1/2}.
\]

If, in addition, (4.18) holds, then
\[
\frac{\hat{\kappa}_\tau}{\kappa_i} \sim \left( \ln \frac{\hat{\kappa}_\eta}{\kappa_i} \right)^{1-\alpha}.
\]
Proof. Apply Proposition 4.5 to obtain (4.19). To prove (4.20), solve for $\hat{c}$ in (4.18) to obtain

$$\hat{c}_{\kappa_0, \infty} = \frac{\hat{\eta}^{2/3}}{\kappa^2}.$$ 

Comparing to (4.11) we find that $\hat{c}_{\kappa_0, \infty} \sim \hat{c}_{2\kappa, \infty}$. By (4.9) and (4.3) we have

$$\hat{c}_{\kappa_0, \infty} \gg \hat{c}_{\kappa_1, \infty}.$$ 

Applying the first relation in (4.1) we have

$$1 \sim \frac{\hat{c}_{2\kappa_1, \infty}}{\hat{c}_{2\kappa_1, \kappa_1}} \sim \frac{\hat{c}_{2\kappa, \kappa_1}}{\hat{c}_{2\kappa_1, \kappa_1}}$$

so that $\hat{c} \sim \hat{c}_{2\kappa_1, \kappa_1}$. From (4.11) and (4.13) we conclude with

$$\tilde{\kappa}^2 = \frac{\tilde{c}}{\hat{c}_{\kappa_0, \infty}} \sim \frac{\hat{c}_{2\kappa_1, \kappa_1}}{\hat{c}_{2\kappa_1, \kappa_1}} \sim \frac{\tilde{\kappa}^2}{2\kappa_1} \left( \ln \frac{\tilde{\kappa}}{\kappa_1} \right)^{1-\alpha}.$$ 

Remark 4.7. Under the conditions (4.12), (4.9), and (4.10) the relations in (4.5) will also hold for small $\theta_1$ and very small $\theta_2$ since we may take

$$\theta_1 \sim \left( \ln \frac{\tilde{\kappa}_\eta}{\kappa_1} \right)^{-1/2} \quad \text{and} \quad \theta_2 \sim \frac{\tilde{\kappa}_\eta}{\kappa_\eta} \left( \ln \frac{\tilde{\kappa}_\eta}{\kappa_1} \right)^{1/2}.$$ 

Thus the assumptions in Theorem 4.6 are consistent.

In Theorem 4.6 (and Proposition 4.5) we showed that the Kraichnan energy spectrum (4.10), together with other assumptions, implies a two-sided relation between $\tilde{\kappa}_\sigma$ and $\tilde{\eta}$, namely (4.19). Our next result shows that (4.19) alone implies one-side of the energy spectrum, and that together with the basic assumptions of the Kraichnan theory, (4.19) implies (4.11) up to a logarithmic term.

Proposition 4.8. If (4.17) holds, then

$$\tilde{c}_{2\kappa, 2\kappa} \lesssim \frac{\eta^{2/3}}{\kappa^2} \ln \frac{\tilde{\kappa}_\eta}{\kappa_1}.$$ 

If, in addition (4.12) and (4.9) hold then

$$\tilde{c}_{2\kappa, \kappa_1} \lesssim \frac{\eta^{2/3}}{\kappa^2} \ln \frac{\tilde{\kappa}_\eta}{\kappa_1}.$$
Proof. From (3.16) we have
\[
\hat{\eta}_{\kappa,2\kappa} \leq \left( \frac{2\kappa}{\tilde{\kappa}_\sigma} \right)^2 \hat{\eta},
\]
so that
\[
\hat{\kappa}_{\kappa,2\kappa} = \frac{1}{L^2} \langle |u_{\kappa,2\kappa}|^2 \rangle \leq \frac{1}{\kappa^4 L^2} \langle |Au_{\kappa,2\kappa}|^2 \rangle \leq \frac{1}{\nu \kappa^4} \hat{n}_{\kappa,2\kappa} \leq \frac{1}{\nu \kappa^4} \hat{n}_{\kappa,2\kappa}
\]
\[
\leq \frac{1}{\nu \kappa^4} \left( \frac{2\kappa}{\tilde{\kappa}_\sigma} \right)^2 \hat{\eta} = \frac{4\hat{\eta}^{2/3}}{\kappa^2} \frac{\hat{\eta}^{1/3}}{\nu \tilde{\kappa}_\sigma^2} = \frac{4\hat{\eta}^{2/3}}{\kappa^2} \left( \frac{\tilde{\kappa}_\eta}{\tilde{\kappa}_\sigma} \right)^2.
\]
Apply (4.5), to obtain (4.21).
To obtain (4.22) sum (4.21) over $2\tilde{\kappa}_\sigma \leq \kappa < 2^n \kappa$ to obtain
\[
\hat{\kappa}_{2\kappa,\theta \tilde{\kappa}_\sigma} = \sum \hat{\kappa}_{\kappa,2\kappa} \lesssim \hat{\eta}^{2/3} \frac{\hat{\kappa}_\eta}{\kappa_i} \ln \frac{\tilde{\kappa}_\eta}{\hat{\kappa}_i} \sum_{j=1}^n \frac{1}{\hat{\eta}^{2/3} \frac{\hat{\kappa}_\eta}{\kappa_i}} \ln \frac{\tilde{\kappa}_\eta}{\hat{\kappa}_i}.
\]
Apply once again (4.17) to complete the estimate
\[
\hat{\kappa}_{\theta \tilde{\kappa}_\sigma,\theta \tilde{\kappa}_\sigma} = \frac{1}{L^2} \langle |u_{\theta \tilde{\kappa}_\sigma,\theta \tilde{\kappa}_\sigma}|^2 \rangle \leq \frac{1}{L^2 (\theta \tilde{\kappa}_\sigma)^4} \langle |Au_{\theta \tilde{\kappa}_\sigma,\theta \tilde{\kappa}_\sigma}|^2 \rangle \leq \frac{1}{\nu (\theta \tilde{\kappa}_\sigma)^4} \frac{\hat{\eta} \ln (\tilde{\kappa}_\eta/\kappa_i)}{\nu \theta^4 \tilde{\kappa}_\sigma^2} \ln \frac{\tilde{\kappa}_\eta}{\hat{\kappa}_i}.
\]
Putting these last two estimates together, we have
\[
\hat{\kappa}_{2\kappa,\theta \tilde{\kappa}_\sigma} \lesssim \frac{\hat{\eta}^{2/3}}{\kappa_i^2} \ln \frac{\tilde{\kappa}_\eta}{\kappa_i} \sim \frac{\hat{\eta}^{2/3}}{\kappa_i^2} \ln \frac{\tilde{\kappa}_\eta}{\kappa_i} \sim \frac{\hat{\eta}^{2/3}}{\kappa_i^2} \ln \frac{\tilde{\kappa}_\eta}{\kappa_i}.
\]

\[\square\]

Proposition 4.9. If, in addition to (4.9), (4.12), and (4.17), we have
\[
\hat{\kappa}_{\kappa,\infty} \sim \hat{\kappa}_{2\kappa,\infty},
\]
then
\[
\hat{\eta} \sim \hat{\kappa}_{\kappa,\infty,\kappa_i}^{3/2} \left( \ln \frac{\tilde{\kappa}_\eta}{\kappa_i} \right)^{3/2}.
\]

Proof. Proceed as in (3.16), then apply (4.17) to reach
\[
\hat{\eta} \geq \hat{\kappa}_{2\kappa,\infty} \geq \hat{\eta} \left[ 1 - \left( \frac{2\kappa_i}{\tilde{\kappa}_\sigma} \right)^2 \right] \sim \hat{\eta} \left[ 1 - \left( \frac{2\kappa_i}{\tilde{\kappa}_\sigma} \right)^2 \ln \frac{\tilde{\kappa}_\eta}{2\kappa_i} \right] \sim \hat{\kappa}_\eta.
\]
By (4.12), we have \( \tilde{\eta}_{2,\kappa,\infty} \gg \tilde{\eta}_{\kappa,\infty} \), so that by Lemma 4.1, we have \( \tilde{e}_{2,\kappa,\kappa} \sim \tilde{e}_{2,\kappa,\infty} \).

Apply Proposition 4.8 to obtain

\[
\tilde{e}_{2,\kappa,\kappa} \lesssim \frac{\tilde{\eta}^{2/3}}{\kappa^2} \ln \frac{\kappa_n}{\kappa},
\]

from which (4.24) follows immediately.

\[ \blacktriangleleft \]

At present our considerations do not seem to provide a rigorous converse of (4.22). However, we can rigorously establish a weak converse of (4.24). For this purpose we define the shape function \( \psi(\kappa) \) of the distribution of \( |u_{\kappa,\infty}| \) by the relation

\[
\langle \langle |u_{\kappa/2,\infty}|^2 \rangle \rangle = \psi(\kappa)^2 \left( \langle |u_{\kappa/2,\infty}|^2 \rangle \right)^2.
\]

**Proposition 4.10.** If (4.17) holds, then

\[
\tilde{e}_{\kappa/2,\infty} \lesssim \frac{\tilde{\eta}^{2/3}}{\kappa^2} \left[ \frac{1}{\psi(\kappa)} \left( \ln \frac{\kappa}{\kappa_0} \right)^{-1/2} \left( \ln \frac{\kappa_n}{\kappa} \right)^{-1/2} \left( \frac{\kappa_0}{\kappa} \right) \right].
\]

**Proof.** By Agmon’s inequality we find

\[
|L^2 \mathbf{e}_\kappa^{\text{ext}}| = |(B(q, q), Ap)| = |(B(q, Ap), q)| \leq c_1 |q|^2 |A^{3/2} p|^{1/2} |A^{5/2} p|^{1/2} \leq c_1 |q|^2 |Ap| \kappa^2,
\]

which (with \( c_5 = 2\pi c_1 \)) follows

\[
|\mathbf{e}_\kappa^{\text{ext}}| \leq \frac{c_5}{\kappa_0} \frac{1}{L^2} |u_{\kappa,\infty}|^2 \frac{1}{L} |Au_{0, \kappa}| \kappa^2 \leq \frac{c_5}{\kappa_0} \frac{1}{L^2} |u_{\kappa/2,\infty}|^2 \frac{1}{L} |Au_{0, \kappa}| \kappa^2.
\]

Apply the Cauchy-Schwarz inequality to obtain

\[
|\langle \mathbf{e}_\kappa^{\text{ext}} \rangle| \leq \frac{c_5}{\kappa_0} \frac{1}{L^2} \langle \langle |u_{\kappa/2,\infty}|^2 \rangle \rangle^{1/2} \langle \frac{|Au_{0, \kappa}|^2}{L^2} \rangle^{1/2} \kappa^2.
\]

Now use the relations in (3.1) in (4.28) to find

\[
|\langle \mathbf{e}_\kappa^{\text{ext}} \rangle| \leq \frac{c_5}{\kappa_0} \psi(\kappa) \tilde{e}_{\kappa,\infty} \left( \frac{\kappa^2 \langle \langle |u_{0,\infty}|^2 \rangle \rangle}{L^2} \right)^{1/2} \kappa^2
\]

\[
\leq \frac{c_5}{\kappa_0} \psi(\kappa) \tilde{e}_{\kappa,\infty} \kappa^3 \left( \frac{\tilde{e}}{\nu} \right)^{1/2}
\]

\[
= \frac{c_5}{\kappa_0} \psi(\kappa) \tilde{e}_{\kappa,\infty} \frac{\kappa^3 \tilde{\eta}^{1/2}}{\nu^{1/2} \kappa_0},
\]
and then, assuming the basic assumption (4.17) holds

$$\frac{|\langle \mathcal{E}_{k}^- \rangle |}{\eta^{1/3}} \leq \frac{c_5}{\kappa_0} \psi(\kappa) \tilde{e}_{k,\infty} \kappa \tilde{\eta} \tilde{\eta} \lesssim \psi(\kappa) \tilde{e}_{k,\infty} \left( \ln \frac{\tilde{\kappa}_{\eta}}{\kappa_0} \right)^{1/2} \kappa^2 \frac{\kappa}{\kappa_0}. $$

Rearranging, we have

$$\tilde{e}_{k,\infty} \gtrsim \frac{|\langle \mathcal{E}_{k}^- \rangle |}{\eta^{1/3}} \left( \ln \frac{\tilde{\kappa}_{\eta}}{\kappa_0} \right)^{-1/2} \frac{1}{\kappa} \frac{\kappa_0}{\kappa^2}. $$

Let

$$p = r_0 + r_{-1} + \cdots + r_{-N} = r_0 + p_{-1}, \quad q = r_1 + r_2 + \cdots,$$

where \(r_n = (P_{2^n \kappa} - P_{2^{n-1} \kappa})u\). In terms of these components, the flux toward larger wavenumbers may be decomposed as

$$L^2 \mathcal{E}^-_{\kappa} = - (B(p, p), Aq) = - (B(r_0 + p_{-1}, r_0 + p_{-1}), A_r_1) = L^2 \mathcal{E}^-_{\kappa_1} + L^2 \mathcal{E}^-_{\kappa_2},$$

where

$$L^2 \mathcal{E}^-_{\kappa_1} = -(B(r_0, r_0), A_r_1)$$

$$L^2 \mathcal{E}^-_{\kappa_2} = -(B(r_0, p_{-1} + B(p_{-1}, r_0), A_r_1).$$

Using Agmon's inequality, \(|A^{1/2} r_0| \leq 2/\kappa |A r_0|\), and \(|A^{1/2} r_0| \leq \kappa |r_0|\), we estimate the first term by

$$L^2 |\mathcal{E}^-_{\kappa_1}| = |(B(r_0, r_0), A_r_1)| \leq \sqrt{2} c_1 |r_0| \|A r_1 \| |A r_0| \leq \sqrt{2} c_1 |r_0| (2 \kappa)^2 |r_1| |A r_0|$$

$$\leq 4 \sqrt{2} c_1 \kappa^2 \frac{|r_0 + r_1|^2}{2} |A r_0| \leq 2 \sqrt{2} c_1 \kappa^2 |u_{\kappa/2,\infty}|^2 |A u_{0,2\kappa}|.$$

Proceeding as from (4.27), we obtain

$$\tilde{e}_{\kappa/2,\infty} \gtrsim \frac{|\langle \mathcal{E}_{\kappa/2}^- \rangle |}{\eta^{1/3}} \frac{1}{\psi(\kappa/2)} \left( \ln \frac{\tilde{\kappa}_{\eta}}{\kappa_0} \right)^{-1/2} \frac{1}{\kappa} \frac{\kappa_0}{\kappa}. $$

The second term \(\mathcal{E}^-_{\kappa_2}\) is estimated differently, first by

$$L^2 |\mathcal{E}^-_{\kappa_2}| \leq c_6 |r_0| \|A p_{-1} \| |A r_1| \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} + c_6 \|p_{-1}\| \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} \|r_0\| |A r_1|$$

$$\leq c_6 |r_0| \frac{\kappa}{2} \|p_{-1}\| \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} (2 \kappa)^2 |r_1| + c_6 \|p_{-1}\| \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} \kappa |r_0| (2 \kappa)^2 |A r_1|$$

$$\leq 6 c_6 \kappa^3 \|p_{-1}\| \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} |r_0| |r_1|$$

$$\leq 3 c_6 \kappa^3 \|u\| \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} \|u_{\kappa/2,\infty}\|^2.$$
Again the relations in \((3.1)\) to find

\[
|\mathbf{E}_{\kappa}^\rightarrow| \leq 3c_6 \left( \frac{\kappa}{\kappa_0} \right) \frac{1}{L^2} \left( \frac{\langle |u_{\kappa/2,\infty}|^2 \rangle}{L} \right)^{1/2} \frac{1}{L} \left( \frac{\langle |u_{\kappa}^2 \rangle^{1/2} \rangle}{L^2} \right)^{1/2} \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} \kappa^2
\]

\[
\leq 3c_6 \frac{\kappa^3}{\kappa_0^2} \psi(\kappa) \frac{\kappa}{\kappa_0} \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} \left( \frac{\hat{\varepsilon}}{\mu} \right)^{1/2}
\]

\[
\leq 3c_6 \frac{\kappa}{\kappa_0} \kappa^2 \psi(\kappa) \frac{\kappa}{\kappa_0} \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} \left( \frac{\hat{\kappa}_\eta}{\hat{\kappa}_{\sigma}} \right) \frac{\kappa^1/3}{\kappa_\eta}
\]

The assumption \((4.17)\) gives us

\[
|\mathbf{E}_{\kappa}^\rightarrow| \lesssim \frac{\kappa}{\kappa_0} \kappa^2 \psi(\kappa) \frac{\kappa}{\kappa_0} \left( \ln \frac{\kappa}{\kappa_0} \right)^{1/2} \left( \ln \frac{\hat{\kappa}_\eta}{\hat{\kappa}_{\sigma}} \right)^{1/2} \frac{\kappa^1/3}{\kappa_\eta},
\]

which, after rearrangement reads as

\[
\tilde{\varepsilon}_{\kappa/2,\infty} \gtrsim \frac{|\mathbf{E}_{\kappa}^\rightarrow|}{\hat{\kappa}_\eta^{1/3} \hat{\kappa}_{\sigma}^2} \frac{1}{\psi(\kappa)} \left( \ln \frac{\kappa}{\kappa_0} \right)^{-1/2} \left( \ln \frac{\hat{\kappa}_\eta}{\hat{\kappa}_{\sigma}} \right)^{-1/2} \left( \frac{\kappa_0}{\kappa} \right) \frac{1}{\kappa^2}.
\]

(4.31)

Since, in the inertial range we have

\[
\hat{\eta} \sim \langle \mathbf{E}_{\kappa}^\rightarrow \rangle - \langle \mathbf{E}_{\kappa/2,\infty}^\rightarrow \rangle \leq |\langle \mathbf{E}_{\kappa}^\rightarrow \rangle| + |\langle \mathbf{E}_{\kappa/2,\infty}^\rightarrow \rangle|
\]

combining \((4.29), (4.30), (4.31)\), we have

\[
\tilde{\varepsilon}_{\kappa/2,\infty} \gtrsim \frac{|\langle \mathbf{E}_{\kappa}^\rightarrow \rangle| + |\langle \mathbf{E}_{\kappa/2,\infty}^\rightarrow \rangle|}{\hat{\kappa}_\eta^{1/3} \hat{\kappa}_{\sigma}^2} \left[ \frac{1}{\psi(\kappa)} \left( \ln \frac{\kappa}{\kappa_0} \right)^{-1/2} \left( \ln \frac{\hat{\kappa}_\eta}{\hat{\kappa}_{\sigma}} \right)^{-1/2} \left( \frac{\kappa_0}{\kappa} \right) \right],
\]

from which follows \((4.26)\).

\[\square\]

**Remark 4.11.** According to heuristic arguments the ratio \(\kappa_0/\kappa\) should be missing in \((4.26)\). We conjecture that this term is parasitic, appearing because the Sobolev-type inequalities (particularly Agmon's inequality) do not distinguish between the wave numbers present in the functions to which they are applied. We also conjecture that the shape function \(\psi\) is nearly constant over the inertial range.

5. **Exponential decay**

In this paragraph we establish that indeed \(\tilde{\varepsilon}_{\kappa,\infty}\) decays exponentially in \(\kappa\). In fact we will establish the finite average analogue of an estimate asserted without proof in [F97].
\section*{5.1 Lemma.} For a certain absolute constant \(c_7\), and a certain \(C_6 = C_6(\kappa/\kappa_0)\) we have for \(G \geq C_6\) that

\[ \|e^{tA^{1/2}}u(t)\| \leq M, \quad \forall \ t \geq t_G \overset{\text{def}}{=} \frac{1}{2c_7\lambda_0 \nu G^2 \ln G} \]

where

\[(5.1) \quad \ell = [2c_7G \ln G]^{-1} \kappa_0^{-1} \quad \text{and} \quad M = \frac{5}{2} \sqrt{\kappa_0 \nu G}.
\]

\textbf{Proof.}

Taking the product of (1.1) with \(A e^{2\alpha t \lambda_0^{1/2}t}\), integrating over \(\Omega\), and letting \(v(t) = e^{\alpha t \lambda_0^{1/2}t} u(t)\), we obtain for a certain \(c_8 \geq 1\)

\[ \frac{1}{2} \frac{d}{dt} \|v\|^2 + \nu |Av|^2 \leq c_8 \|v\|^3 \lambda_0^{1/2} \left( \frac{\|Av\|^2}{\lambda_0 \|v\|^2} + 1 \right)^{1/2} + e^{\pi \alpha} |Av||f| + \alpha |Av||v|,
\]

where for the nonlinear term we used the approach in [FT] as well as the estimate (1.9).

Use Young’s inequality (twice) to find

\[ \frac{1}{2} \frac{d}{dt} \|v\|^2 + \nu |Av|^2 \leq c_8 \|v\|^3 \lambda_0^{1/2} \left( \frac{\|Av\|^2}{\lambda_0 \|v\|^2} + 1 \right)^{1/2} + \frac{e^{2\pi \alpha}}{\nu} |f|^2 + \frac{\alpha^2}{\nu} \|v\|^2.
\]

Now split the second term on the left hand side and set \(\|Av\|^2 = \lambda_0 \|v\|^2 \xi\) to reach

\[(5.2) \quad \frac{d}{dt} \|v\|^2 + \frac{\nu}{2} |Av|^2 \leq -\frac{\nu}{2} \lambda_0 \|v\|^2 \xi + 2c_8 \|v\|^3 \lambda_0^{1/2} \xi^{1/2} |\ln (\xi + 1)|^{1/2} + \frac{2e^{2\pi \alpha}}{\nu} |f|^2 + \frac{2\alpha^2}{\nu} \|v\|^2
\]

\[ \leq \frac{1}{2} \|v\|^2 \lambda_0 \nu g(\xi) + \frac{2e^{2\pi \alpha}}{\nu} |f|^2 + \frac{2\alpha^2}{\nu} \|v\|^2.
\]

where

\[ g(\xi) = \beta \xi^{1/2} |\ln (\xi + 1)|^{1/2} - \xi \quad \text{and} \quad \beta = \frac{4c_8 \|v\|}{\lambda_0^{1/2} \nu}.
\]

It follows from elementary calculations that

\[ g''(\xi) \leq 0, \quad \text{for} \ \xi \geq 0,
\]

\[ g'(1) \leq 0 \quad \text{if and only if} \quad \beta \leq \beta_0 \overset{\text{def}}{=} \frac{4 (\ln 2)^{1/2}}{2 \ln 2 + 1},
\]

\[(5.3) \quad \beta \leq \beta_0 \overset{\text{def}}{=} (\ln 2)^{-1/2} \implies g(\xi) \leq 0 \ \forall \ \xi \geq 1,
\]
and

\( g(\xi) \leq 0 \text{ if } \xi \geq \xi_\beta \overset{\text{def}}{=} 6\beta^2 (1 + \ln \beta) . \)

Thus if \( \beta > \beta_{00} \), then \( \max_{\xi \geq 1} g(\xi) > 0 \), and there exists \( \xi_m \geq 1 \) such that \( g(\xi_m) = \max_{\xi \geq 1} g(\xi) \). From (5.4) we have

\[
g(\xi_m) \leq \beta \xi_m^{1/2} [\ln(\xi_m + 1)]^{1/2} \leq \beta \xi_\beta^{1/2} [\ln(\xi_\beta + 1)]^{1/2} \leq \xi_\beta^2 = 6\beta^2 \ln(\epsilon^2) .
\]

Combining this with (5.2) we have for all \( \beta \geq 0 \) that

\(\frac{d}{dt} \|v\|^2 + \frac{\nu}{2} |Av|^2 \leq \frac{2ce^{2\pi\alpha t}}{\nu} \|f\|^2 + \frac{2\alpha^2}{\nu} \|v\|^2 + 24c_8^2 \|v\|^4 \ln \left( \frac{(4ec_8)^2 \|v\|^2}{\lambda_0 \nu^2} \right),\)

or, in terms of

\[
\varphi = \frac{(4ec_8)^2 \|v\|^2}{\lambda_0 \nu^2},
\]

\[
\frac{d\varphi}{dt} + \frac{(4ec_8)^2}{2\lambda_0 \nu^2} |Av|^2 \leq c_9 \lambda_0 \nu \varphi^2 \ln \varphi + c_{10} \frac{e^{2\pi\alpha t}}{\lambda_0 \nu^3} \|f\|^2 + \frac{2\alpha^2}{\nu} \varphi
\]

\[
\leq c_9 \lambda_0 \nu \varphi^2 \ln \varphi + c_{10} \frac{e^{2\pi\alpha t}}{\lambda_0 \nu^3} \|f\|^2 + \frac{\alpha^4}{c_9 \lambda_0 \nu^3} \varphi^2
\]

\[
\leq 2c_9 \lambda_0 \nu \varphi^2 \ln(\varphi + e) + c_{10} \frac{e^{2\pi\alpha t}}{\lambda_0 \nu^3} \|f\|^2 + \frac{\alpha^4}{c_9 \lambda_0 \nu^3},
\]

where \( 1/5 < c_9 = 3/(2e^2) < 1/4 \). Dropping the second term on the left hand side, we have

\[
\frac{d\varphi}{dt} \leq \frac{1}{2} \lambda_0 \nu \varphi^2 \ln(\varphi + e) + c_{10} e^{2\pi\alpha t} \lambda_0 \nu G^2 + \frac{5\alpha^4}{\lambda_0 \nu^3}.
\]

Set

\[
\delta = \frac{\alpha}{\nu \lambda_0^{1/2}}, \quad \tilde{\varphi}(\tau) = \varphi \left( \frac{\tau}{\lambda_0 \nu} \right)
\]

so that

\[
\frac{d\tilde{\varphi}}{d\tau} \leq \frac{1}{2} \tilde{\varphi}^2 \ln(\tilde{\varphi} + e) + c_{10} e^{2\pi\alpha_0 \delta \tau} G^2 + 5\delta^4
\]

Now set \( \delta = G \) (this fixes \( \alpha \)). Then for \( \tau \leq G^{-1} \) and \( C_5 = c_{10} e^{2\pi \alpha_0} \) we have

\[
\frac{d\tilde{\varphi}}{d\tau} \leq \tilde{\varphi}^2 \ln(\tilde{\varphi} + e) + C_5 G^2 + 5G^4.
\]

If \( u(0) \) is in \( B \), then

\[
\tilde{\varphi}(0) = \frac{(4ec_8)^2}{\lambda_0 \nu^2} \|u(0)\|^2 \leq (8ec_8)^2 G^2,
\]
and as long as
\[ \tilde{\varphi}(\tau) = \frac{(4cc_8)^2}{\lambda_0\nu^2} \| v(\tau) \|^2 \leq 2e(8cc_8)^2G^2 = c_7G^2 \]
we have
\[ \frac{d\tilde{\varphi}}{d\tau} \leq \frac{1}{2} c_7G^2 \ln(c_7G^2 + e)\tilde{\varphi} + C_5G^2 + 5G^4. \]
Since \( c_7G^2 + e \leq G^4 \) for \( G \geq c_{11} \) we have for \( u(0) \in B \) that
\[
\tilde{\varphi}(\tau) \leq e^{r[c_7G^2 \ln(c_7G^2 + e)]/2} \left[ \tilde{\varphi}(0) + \frac{2(C_5 + 5G^2)}{c_7 \ln(c_7G^2 + e)} \right] \\
\leq e^{r[2c_7G^2 \ln G]} \left[ (8cc_8)^2G^2 + \frac{2(C_5 + 5G^2)}{c_7 \ln(c_7G^2 + e)} \right].
\]
For large enough \( C_6 > c_{11} \) we have that \( G \geq C_6 \) implies
\[
(5.7) \quad \frac{2(C_5/G^2 + 5)}{c_7 \ln(c_7G^2 + e)} \leq (6cc_8)^2.
\]
We then have that
\[
(5.8) \quad \tilde{\varphi}(\tau) \leq c_{12}G^2 \quad \text{for} \quad 0 \leq \tau \leq \tau_G \quad \text{def} \quad \frac{1}{2c_7G^2 \ln G} = \lambda_0 \nu t_G.
\]
where \( c_{12} = e(10cc_8)^2 \). In particular, we have
\[
\| e^{[2c_7G \ln G]^{-1} \kappa_0^{-1} A^{1/2}} u(t_G) \| = \frac{\lambda_0\nu^2}{(4cc_8)^2} \varphi(t_G) = \frac{\lambda_0\nu^2}{(4cc_8)^2} \tilde{\varphi}(\tau_G) \leq \left( \frac{5}{2} \right)^2 e\lambda_0\nu^2G^2,
\]
i.e. \( \| e^{tA^{1/2}} u(t_G) \| \leq M. \) Since \( u(t_0) \in B \) for all \( t_0 \geq 0 \) we also have
\[
\| e^{tA^{1/2}} u(t_0 + t_G) \| \leq M,
\]
and hence
\[
\| e^{tA^{1/2}} u(t) \| \leq M, \quad \forall t \geq t_G.
\]

5.2 Theorem. There exists \( C_7 = C_7(\kappa/\kappa_0) \) and an absolute constant \( c_{13} \) such that for \( G \geq \max\{C_6, c_{12}\} \),
\[
(5.9) \quad t_0 \geq t_G, \quad \text{and the averaging time} \quad T \geq (\nu\kappa_0^2)^{-1}
\]
we have
\[
(5.10) \quad \dot{\kappa}_\infty \leq \left[ C_7 + c_{13}G^2 \ln G \right] G^2\nu^2\kappa_0^2 \left( \frac{\kappa_0}{\kappa} \right)^4 e^{-2t\kappa}.
\]
Proof. Change the definition of $v(t)$ to
\[ v(t) = e^{tA^{1/2}} u(t) \quad \text{for all} \quad t \geq t_G. \]

Then this $v(t)$ satisfies slightly modified versions of (5.5) and (5.6), namely
\[
\frac{d}{dt} \|v\|^2 + \frac{\nu}{2} |Av|^2 \leq \frac{1}{2} \|v\|^2 \lambda_0 \nu g(\xi) + \frac{e^{2\pi\alpha t}}{\nu} |f|^2
\]
and
\[
(5.11) \quad \frac{d\varphi}{dt} + \frac{(4c_8)^2}{2\lambda_0 \nu} |Av|^2 \leq c_9 \lambda_0 \nu \varphi^2 \ln \varphi + \frac{c_{10} e^{2\pi t}}{2\lambda_0 \nu} |f|^2.
\]

Integrating (5.11) from $t_0 \geq t_G$ to $t_0 + T$, and applying (5.8) yields
\[
(\frac{(4c_8)^2}{2\lambda_0 \nu}) \int_{t_0}^{t_0 + T} |Av|^2 dt \leq c_{12} G^2 + \frac{c_{12}}{4} T \lambda_0 \nu G^4 \ln(c_{12} G^2) + C_8 T \lambda_0 \nu G^2,
\]
where
\[
C_8 = \frac{c_{10} e^{2\pi t}}{2}.
\]

Since $G \geq c_{12}$ we have for $T \geq (\lambda_0 \nu)^{-1}$ that
\[
(5.12) \quad \langle |Av|^2 \rangle \leq C_9 (\lambda_0 \nu)^2 G^2 + c_{14} (\lambda_0 \nu)^2 G^4 \ln G.
\]

On the other hand
\[
\frac{1}{L^2} \langle |Av(t)|^2 \rangle = \frac{1}{L^2} \langle |Ae^{tA^{1/2}} u(t)|^2 \rangle
\]
\[
= \langle \sum_{k \neq 0} \left( \frac{2\pi |k|}{L} \right)^4 e^{2t|k|^2 / L} |\hat{u}_k(t)|^2 \rangle
\]
\[
\geq \langle \sum_{2\pi |k| / L \geq \kappa} \left( \frac{2\pi |k|}{L} \right)^4 e^{2t|k|^2 / L} |\hat{u}_k(t)|^2 \rangle
\]
\[
\geq \kappa^4 e^{2t\kappa} \langle \sum_{2\pi |k| / L \geq \kappa} |\hat{u}_k(t)|^2 \rangle
\]
\[
= \kappa^4 e^{2t\kappa} \ell_{\kappa, \infty},
\]
which when combined with (5.12) yields (5.10).
5.3 Remark.

(i) A basic estimate concerning the radius of convergence of the Taylor expansion in the spatial variable of any $u$ on the global attractor established in [Ku] suggests that the exponent $\ell$ in (5.10) may actually be less than $G^{-1/2}(\ln G)^{-1/4}$. Our method is not strong enough to lead to this result.

(ii) The estimate suggested by the heuristic assumption $\tilde{c}_{\kappa, \infty} \ll \tilde{c}_{\kappa_0, \infty}$ is much stronger than both (5.10) and the possible variant discussed in (i), at least for scales $\kappa$ comparable to $\tilde{\kappa}_\eta$.

5.4 Corollary. Under the assumptions of Proposition 3.1 and Theorem 5.2 we have for $G \geq C_{10} = C_{10}(\kappa/\kappa_0)$ that

\[
\tilde{\eta}_{\kappa, \infty} - \tilde{\eta} e^{-\kappa G/\kappa_0(\ln G)^{-1}}
\]

for $\kappa$ satisfying

\[
\frac{\kappa}{\kappa_0} \geq 20c_7G(\ln G)^2.
\]

Proof. It follows from (3.5) that

\[
G \leq C_2 \left( \frac{(\tilde{\eta}/\nu^3)^{1/6}}{\kappa_0} \right) \leq C_2 \frac{\tilde{\eta}}{\kappa_0^{6/\nu^3}}.
\]

Thus for any $u(0) \in B$ we have by (5.12) that for the averaging time needed for both Proposition 3.1 and Theorem 5.2

\[
e^{\ell_\kappa \tilde{\eta}_{\kappa, \infty}} \leq \frac{\nu}{L^2} \langle |e^{\ell A}Au_{\kappa, \infty}|^2 \rangle \leq \frac{\nu}{L^2} \langle |e^{\ell A}Au|^2 \rangle \leq \frac{\nu}{L^2} \langle |Av|^2 \rangle
\]

\[
\leq \frac{\nu}{L^2} [C_9 + c_{14}G^2 \ln G] G \lambda_0^2 \nu^2 = C_2 \frac{C_2}{\kappa_0^6 \nu^3} \tilde{\eta}
\]

\[
\leq C_{11}[1 + G^2 \ln G]G\tilde{\eta}
\]

We want

\[
e^{-\kappa \ell/2}C_{11}[1 + G^2 \ln G]G \leq 1,
\]

which, for $G \geq e$, follows from

\[
e^{-\kappa \ell/2}2C_{11}G^3 \ln G \leq 1,
\]

or equivalently

\[
\kappa \geq \frac{2}{\ell} \ln(2C_{11}G^3 \ln G) = 4c_7G \ln(G)\kappa_0 \ln(2C_{11}G^3 \ln G),
\]
For $G \geq C_{10} = \max\{2C_{11}, e, C_6, c_{12}\}$ the last term in (5.17) is bounded from above by $20c_7\kappa_0 G (\ln G)^2$. Thus (5.16) and hence (5.15) follow from (5.14).

Consequently, if (5.14) holds, then for $\kappa$ as in (5.14)

$$\tilde{\eta}_{\kappa, \infty} \leq e^{-\kappa t/2}\tilde{\eta}$$

from which (5.13) follows.

We can reformulate 5.4 entirely in terms of $\tilde{\kappa}_\eta$, i.e. without any reference to $G$ (except for the mild assumption (3.3)).

**5.5 Corollary.** If the assumptions of Proposition 3.1 hold together with (5.9), we have

$$\tilde{\eta}_{\kappa, \infty} \leq \tilde{\eta} \exp \left[-C_{12} \left(\kappa \left(\frac{\kappa}{\tilde{\kappa}_\eta}\right) \left(\frac{\kappa_0}{\tilde{\kappa}_\eta}\right)^5 \frac{1}{\ln(\tilde{\kappa}_\eta/\kappa_0)}\right)\right].$$

for $\kappa$ satisfying

$$\frac{\kappa}{\tilde{\kappa}_\eta} \geq C_{13} \left(\frac{\tilde{\kappa}_\eta}{\kappa_0}\right)^5 \left(\frac{\ln(\tilde{\kappa}_\eta)}{\kappa_0}\right)^2, \text{ provided } \frac{\tilde{\kappa}_\eta}{\kappa_0} \geq \max\{C_{10}^{1/3}, C_2^{2/3}\}.$$

**Proof.** From (5.19) and the lower bound in (3.5) we have

$$\frac{\kappa}{\kappa_0} \geq C_{13} \left(\frac{\tilde{\kappa}_\eta}{\kappa_0}\right)^6 \left\{\ln(\frac{\tilde{\kappa}_\eta}{\kappa_0})\right\}^2 \geq \frac{C_{13}}{C_2} G \left\{\ln\left(G/C_2\right)^{1/6}\right\}^2 \geq \frac{C_{13}}{144C_2} G (\ln G)^2$$

for $G$ satisfying $G \geq C_2^2$, which by the upper bound in (3.5), is implied by the second condition in (5.19), namely $(\tilde{\kappa}_\eta/\kappa_0)^{3/2} \geq C_2$. Thus (5.19) implies (5.14) for $C_{13} = 2880c_7C_2$ so that (5.13) holds. Similarly we have

$$\frac{1}{G \ln G} \geq \left(\frac{\kappa_0}{\tilde{\kappa}_\eta}\right)^6 \frac{1}{C_2} \left\{\ln\left(C_2\left(\frac{\tilde{\kappa}_\eta}{\kappa_0}\right)^6\right)\right\}^{-1} \geq \left(\frac{\kappa_0}{\tilde{\kappa}_\eta}\right)^6 \frac{2}{15C_2} \left\{\ln\left(\frac{\tilde{\kappa}_\eta}{\kappa_0}\right)\right\}^{-1},$$

Combining (5.20) with (5.13) gives (5.18).

**5.6 Remark.** Note that from (5.18) it follows that $\tilde{\eta}_{\kappa, \infty} \ll \tilde{\eta}$ once (5.19) is satisfied. Unfortunately the rigorous estimate above does not give any relevant information in the “large” portion of the dissipative range between $\tilde{\kappa}_\eta$ and $(\tilde{\kappa}_\eta/\kappa_0)^5\tilde{\kappa}_\eta$. Clearly, one must come with an innovative method in order to obtain a rigorous estimate in that range.
6. Concluding remarks

In this paper we have shown how elements of classical Kolmogorov-Batchelor-Kraichnan turbulence theory emerge in a rigorous way from finite time averages of the physical variables involving the solutions of the Navier-Stokes equations. It seems that in so far as the turbulence issues examined here are concerned, it is not necessary to restrict the flow to the global attractor; it is sufficient to simply consider solutions which have been in the absorbing ball for an adequate, but finite length of time.

The basic problems left open by our analysis are to characterize the driving forces for which $\tilde{\kappa}_e$ and $\tilde{\kappa}_\eta$ are comparable (up to a logarithmic correction) and $\tilde{\eta}_{\kappa_{\eta},\infty}$ is negligible versus $\tilde{\eta}$. Other open problems are explicitly or implicitly suggested in the numbered remarks. We stress that each of these open problems has a rigorous mathematical framework.

The 3-D variant of this paper, i.e. the finite time average version of [FMRT2] (which is the 3-D version of [FJMR]) will be given in [FJRT].
References


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3 This work, though completed after O.P.M. passed away, was initiated by his insistence that the physical content of our paper [FJMR] be made independent of the esoteric Hahn-Banach extension of the classical concept of a limit.