1 Introduction

2 Riemannian Manifolds

This section begins with the definition of a Riemannian Manifold, the basic object of study in Differential Geometry.

2.1 Riemannian Metric

Definition: A Riemannian Manifold is a smooth manifold $M^n$ with a symmetric positive definite inner product $g$ on the tangent spaces $T_m M$ which varies smoothly with $m \in M$.

Thus:

1. $g(X_m + Y_m, Z_m) = g(X_m, Z_m) + g(Y_m, Z_m)$ for $X_m, Y_m, Z_m \in T_m M$.
2. $g(X_m, Y_m) = g(Y_m, X_m)$
3. $g(X_m, X_m) \geq 0$ and equals 0 iff $X_m = 0$.

Smoothness means that if $X, Y$ are smooth vector fields, then the function

$$g(X, Y) : M \rightarrow \mathbb{R}$$

given by $g(X, Y)(m) = g(X_m, Y_m)$ is smooth.

A “slick” definition is that $g$ is a smooth section of the bundle $\text{Sym}^2(T^* M)$ such that pointwise, $g$ lies in the cone of positive definite non-singular symmetric 2-tensors. In particular, the space of all metrics on $M$ is a contractible convex set. This fact is sometimes used to show that some particular differential-geometric invariant of a manifold is actually an invariant of the smooth structure alone.

The vector bundle of $p$-forms on $M$ is the vector bundle, denoted $\bigwedge^p T^* M$, whose fiber over a point $m$ is the alternating multilinear maps $\omega : T_m M \times \cdots \times T_m M \rightarrow \mathbb{R}$. Here alternating means that precomposing $\omega$ by a permutation of the entries changes the output by the sign of that permutation. The vector space of differential $p$-forms is the space $\Omega^p$ of smooth sections of $\bigwedge^p T^* M$.

2.2 Volume Form
An oriented Riemannian manifold \((M^n, g)\) has a Volume Form,
\[
dvol \in \Omega^n(T^*M).
\]
This form can be expressed in an oriented coordinate system \((x_1, \cdots, x_n)\) by the formula:
\[
dvol = \sqrt{\det(g_{i,j})} \, dx_1 \wedge \cdots \wedge dx_n
\]
where \(g_{i,j} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\). In particular this is a nowhere-vanishing form. Thus a metric provides us with a well-defined nonzero section (whose value at a point \(m\) we denote also by just \(dvol\)) of the trivial line bundle \(\bigwedge^n T^*M^n\).

A metric \(g\) on \(T_xM\) determines a metric \(g^*\) on the cotangent bundle \(T^*M\) via the isomorphism:
\[
\psi : T_xM \rightarrow T^*M
\]
which takes \(X\) to \(g(X, \cdot)\). So \(g^*\) is the inner product on \(T^*M\) given by
\[
g^*(\omega, \theta) = g(\psi^{-1}(\omega), \psi^{-1}(\theta)) \text{ for } \omega, \theta \in T^*M.
\]

**Exercise 2.1:** If \(g^{i,j} = g^*(dx_i, dx_j)\) and \(g_{i,j} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\), then \((g^{i,j}) = (g_{i,j})^{-1}\).

If \(e_1, \cdots, e_n\) are local sections of \(T^*M\) which are pointwise orthonormal with respect to \(g^*\) and compatible with the orientation of \(M\), then:
\[
dvol = e_1 \wedge \cdots \wedge e_n.
\]

Some “local” formulas for differential forms are better expressed in terms of a coordinate system, others in terms of local orthonormal sections. It is not possible in general to choose a coordinate system \((x_1, \cdots, x_n)\) so that the 1-forms \(dx_1, \cdots, dx_n\) are orthonormal in the entire coordinate neighborhood, (Riemann showed in his thesis that this is equivalent to the Riemannian connection being flat) although coordinates can be chosen so that the \(dx_i\) are orthonormal at one point.

The metric on \(T_xM\) can be used to define one on all the tensor bundles associated to \(T_xM\). We will often use the inner product in the bundle \(\bigwedge^p T^*M\) which is defined by
\[
\langle dx_{i_1} \wedge \cdots \wedge dx_{i_p}, dx_{j_1} \wedge \cdots \wedge dx_{j_p} \rangle = \prod_k \langle dx_{i_k}, dx_{j_k} \rangle = \prod_k g^{i_k,j_k}
\]
When studying tangent vectors or forms, $\langle \ , \ \rangle$ will denote either $g$ or $g^\ast$ depending on the context.

### 2.3 The Hodge $\ast$ Operator

Poincaré duality is expressed on the level of differential forms using the Hodge $\ast$ operator.

For $(M^n, g)$ an oriented Riemannian Manifold, let

$$\ast : \bigwedge^p (T_m^\ast M) \rightarrow \bigwedge^{n-p} (T_m^\ast M)$$

be the unique linear isomorphism satisfying the formula:

$$\omega \wedge \ast \theta = \langle \omega, \theta \rangle d\text{vol}$$

for each $\omega \in \bigwedge^p (T_m^\ast M)$. Since $\langle \ , \ \rangle$ is a non-degenerate bilinear form, this formula determines $\ast$.

This induces a map (also denoted $\ast$) on the differential forms

$$\ast : \Omega^p \rightarrow \Omega^{n-p}.$$

For example, if $e_1, \ldots, e_n$ is a local oriented orthonormal collection of 1-forms (an oriented orthogonal coframe), then:

$$\ast(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \pm e_{j_1} \wedge \cdots \wedge e_{j_p},$$

where $(i_1, \ldots, i_p, j_1, \ldots, j_p)$ is a $\pm 1$ permutation of $(1, 2, \ldots, n)$.

#### Exercise 2.2:

1. $\ast^2 : \bigwedge^p (T_m^\ast M) \rightarrow \bigwedge^p (T_m^\ast M)$ is given by
   $$\ast^2 \omega = (-1)^p(n-p) \omega.$$

2. $\langle \tau, \omega \rangle = \langle \ast \tau, \ast \omega \rangle$, i.e. $\ast$ is an isometry.

### 2.4 Self-Dual and Anti-Self Dual 2-Forms on a 4-Manifold

From Exercise 2.2 we conclude that on a riemannian 4-manifold $(M^4, g)$, $\ast : \bigwedge^2 (T_m^\ast M) \rightarrow \bigwedge^2 (T_m^\ast M)$ satisfies $\ast^2 = 1$. Thus we have the Orthogonal (check this!) splitting: $\bigwedge^2 (T_m^\ast M) = \bigwedge^+ (T_m^\ast M) \oplus \bigwedge^- (T_m^\ast M)$, where the $\bigwedge^\pm (T_m^\ast M)$ are the $\pm 1$ eigenspaces of $\ast$. This splitting depends on the Riemannian metric, since $\ast$ does. We define $\Omega^\pm$ to be the the space of smooth sections of $\bigwedge^\pm (T_m^\ast M)$. 
Exercise 2.3: If \( e_1, \ldots, e_4 \) is a local oriented orthogonal coframe on \((M^4, g)\), then:

\[
\{ e_1 \wedge e_2 + e_3 \wedge e_4, \ e_1 \wedge e_3 - e_2 \wedge e_4, \ e_1 \wedge e_4 + e_2 \wedge e_3 \}
\]
is a local basis of sections for \( \Omega^2_+ \), and

\[
\{ e_1 \wedge e_2 - e_3 \wedge e_4, \ e_1 \wedge e_3 + e_2 \wedge e_4, \ e_1 \wedge e_4 - e_2 \wedge e_3 \}
\]
is a local basis of sections for \( \Omega^2_- \).

Forms in \( \Omega^2_+ \) are called self dual; forms in \( \Omega^2_- \) are called anti-self dual. There are orthogonal projections:

\[
P_- : \Omega^2 \rightarrow \Omega^2_-, \quad P_-(\omega) = \frac{1}{2}(\omega - *\omega)
\]

\[
P_+ : \Omega^2 \rightarrow \Omega^2_+, \quad P_+(\omega) = \frac{1}{2}(\omega + *\omega)
\]

Exercise 2.4: Use the Hodge Theorem to prove that if \( \beta_+ \) is the dimension of the harmonic self-dual 2-forms and \( \beta_- \) is the dimension of the harmonic anti-self-dual 2-forms, then the signature of \( M \) is \( \beta_+ - \beta_- \).

3 Bundles

3.1 Principal Bundles

We recall the definition and basic examples of principal bundles. I have tried to adhere to the conventions set forth in [KN], so that the structure group acts on the right.

Let \( G \) be a lie group. A (smooth) principal \( G \) bundle over \( M \) is a smooth manifold \( P \) with a free (right) \( G \) action with orbit space \( M \) such that the orbit map \( \pi : P \rightarrow M \) is a locally trivial fiber bundle. Thus, \( M \) is covered by open sets \( U_\alpha \); diffeomorphisms \( \psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G \) are given so that:

1. \( \psi_\alpha \) is equivariant: \( \psi_\alpha(u \cdot g) = \psi_\alpha(u) \cdot g \)
2. \( \pi(\psi^{-1}(u, g)) = u. \)
3. \( (U_\alpha \cap U_\beta) \times G \xrightarrow{\psi_{\alpha\beta}^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\psi_{\beta\alpha}} (U_\alpha \cap U_\beta) \times G \)

is given by \( (u, g) \mapsto (u, gh_{\alpha\beta}) \) for some \( h_{\alpha\beta} \in G \).
**Notation** For \( g \in G \), we write \( R_g \) for right translation and \( L_g \) for left translation by \( g \) either in \( G \) or in \( P \). Also \( g \) will denote the Lie algebra of \( G \). We call \( G \) the **structure group**.

**Examples**

A. The **Trivial Bundle** \( P \cong M \times G \longrightarrow M \). If \( s : M \longrightarrow P \) is a section of a principal \( G \) bundle \( \pi : P \longrightarrow M \), then \( s \) determines a trivialization \( P \cong M \times G \) by \( p \mapsto (\pi(p), g_p) \) where \( g_p \in G \) is determined by the formula \( p \cdot g_p = s(\pi(p)) \). Fixing this section, all others correspond to maps \( h : M \longrightarrow G \) using \( s_h(m) = s(m)h(m) \).

B. The **Frame Bundle** \( GL(T^*M) \longrightarrow M \) is the principal \( GL_n(\mathbb{R}) \) bundle whose fiber over a point \( m \in M \) is the collection of all bases for \( T_m M \); this fiber is (not canonically) isomorphic to \( GL_n(\mathbb{R}) \).

C. The **Orthogonal Frame Bundle** \( O(T^*M) \longrightarrow M \). Given a Riemannian metric on \( M \), we can consider those frames which are orthonormal. This forms a sub-principal bundle of \( GL(T^*M) \) whose fiber is \( O_n \subset GL_n(\mathbb{R}) \).

**Exercise 3.1:** If \( M \) is orientable, then \( O(T^*M) \) is a union of two components. A choice of orientation for \( M \) is the same as a choice of a component of \( O(T^*M) \). This component is then a principal \( SO_n \) bundle, the **Oriented Orthogonal Frame Bundle**.

D. If \( \tilde{M} \longrightarrow M \) is a regular covering space with covering group \( G \), then \( \tilde{M} \longrightarrow M \) is a principal bundle with (discrete) structure group \( G \).

E. If \( E \longrightarrow M \) is an \( \mathbb{R}^k \) (resp. \( \mathbb{C}^k \)) vector bundle over \( M \), then one can construct a principal \( GL_k(\mathbb{R}) \) (resp. \( GL_k(\mathbb{C}) \)) bundle over \( M \) by considering linearly independent bases in the vector space fibers just as in example B. If \( E \) is equipped with a fiberwise positive definite symmetric (resp. Hermitean) inner product, then as in example C one can construct a principal \( O_k \) (resp. \( U_k \)) bundle from \( E \).

### 3.2 Classifying Spaces and Characteristic Classes

The following theorem is well known to Topologists:
**Theorem** Given a (nice) Lie group $G$, there exists a space $BG$ and a principal $G$ bundle $EG\to BG$ so that the correspondence:

$\{\text{Homotopy classes of } f : M \to BG\} \leftrightarrow \{\text{Isomorphism classes of principal } G \text{ bundles over } M\}$

given by

$$(f : M \to BG) \mapsto (f^*(EG) \to M)$$

is a bijection.

(For a proof see e.g. Husemoller Fiber Bundles.)

We call $EG\to BG$ the **Universal $G$ bundle**; $BG$ is called the **Classifying Space** for $G$. The space $EG$ is contractible, and in fact it is easy to prove this theorem once one finds a contractible space on which $G$ acts freely.

The cohomology of these universal bundles are the source of many invariants in bundle theory. We define **characteristic classes** to be the pullbacks of cohomology classes in $BG$. More precisely, if $P \to M$ is a principal $G$ bundle classified by a map $f_P : M \to BG$, and if $\alpha \in H^*(BG)$ is a cohomology class, define $\alpha(P) = f_P^*(\alpha)$.

The class $\alpha(P)$ is obviously an invariant of the isomorphism class of the principal $G$ bundle $P$, and is useful for distinguishing the various bundles over $M$ (e.g. see the next exercise.)

The next theorem provides some sample calculations:

**Theorem**

(a) $H^*(BU_1, \mathbb{Z}) = \mathbb{Z}[c_1], |c_1| = 2$

(b) $H^*(BSO_3, \mathbb{Q}) = \mathbb{Q}[p_1], |p_1| = 4$

(c) $H^*(BSU_2, \mathbb{Z}) = \mathbb{Z}[c_2], |c_2| = 4$

(d) $H^*(BU_n, \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n], |c_k| = 2k$

(e) $H^*(BSO_{2n-1}, \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_n], |p_k| = 4k$

(f) $H^*(BSO_n, \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, \ldots, w_n], |w_k| = k$

In particular, if $P \to M$ is a principal $SO_3$ bundle, then $p_1(P) \in H^4(M, \mathbb{Q})$ is called the **first Pontrjagin class** of $P$. If $P \to M$ is an $SU_2$ bundle, then $c_2(P) \in H^4(M, \mathbb{Z})$ is called the **Second Chern class** of $P$. The $\mathbb{Z}_2$ characteristic classes for $BSO_n$ are called **Stiefel-Whitney classes**.

**Exercise 3.2:** Using the fact that $BSU(2)$ is just the infinite quaternionic projective space $\mathbb{HP}^{\infty}$, show that principal $SU(2)$ bundles over an oriented 4 manifold are determined up to isomorphism by their second Chern class.
Suppose $H \to G$ is a homomorphism of lie groups and $P \to M$ is a principal $G$ bundle. We say that $P$ lifts to a principal $H$ bundle if there is a principal $H$ bundle $Q \to M$ and an $H$ equivariant map $Q \to P$ covering the identity map of $M$. Here $H$ acts on $P$ via the homomorphism $H \to G$. Examples B and C of Section 3.1 shows how to use a Riemannian metric on $M$ to lift the $GL_n$ frame bundle to an $O_n$ bundle.

Lifting can also be stated in terms of classifying spaces. The homomorphism $H \to G$ induces a fibration $BH \to BG$, with fiber $BK$ where $K = \text{Ker}H \to G$. If $f : M \to BG$ classifies the bundle $P$, then $P$ lifts to a principal $H$ bundle iff $f$ lifts to $BH$. Thus the machinery of obstruction theory applies.

**Exercise 3.3:** Using obstruction theory (and a little bit more), show that if $\text{Spin}_n \to \text{SO}_n$ is the 2-fold cover, then an $\text{SO}_n$ bundle $P$ lifts to a $\text{Spin}_n$ bundle iff $w_2(P) = 0$.

### 3.3 Associated Vector Bundles

Let $P \to M$ be a principal $G$ bundle and $V$ a representation of $G$. Thus we are given a homomorphism $\rho : G \to GL(V)$. We are thinking of $GL(V)$ acting on the left of $V$. Let:

$$P \times_{\rho} V = \frac{P \times V}{(p, v) \sim (pg, \rho(g^{-1})v)}$$

**Exercise 3.4:** The function $P \times_{\rho} V \to M$ given by $(p, v) \mapsto \pi(p)$ is well defined and makes $P \times_{\rho} V$ into a vector bundle over $M$ with fiber $V$. We call $E = P \times_{\rho} V$ the vector bundle associated to $P$ by the representation $V$.

**Examples**

A. Take $\mathbb{R}^n$ to be the usual (standard) $\text{SO}_n$ representation. Then the associated vector bundle to a principal $\text{SO}_n$ bundle is an $\mathbb{R}^n$ vector bundle. Similarly we can take the standard representation of $U(n)$ on $\mathbb{C}^n$.

B. The differential at the identity of the map $\text{Ad}_g : G \to G$ given by $h \mapsto ghg^{-1}$ is a linear map $\text{ad}_g : g \to g$. This defines the Adjoint Representation $\text{ad} : G \to GL(g)$. The resulting vector bundle is called the Adjoint bundle and is denoted by $\text{ad} g \to M$.

It is well known that if $G \subset GL_n(\mathbb{C})$, then $g \subset gl_n(\mathbb{C}) = M_{n \times n}(\mathbb{C})$ and $\text{ad}_g(A) = gAg^{-1}$ for $g \in G, A \in g$. 
C. Let $O(T,M) \to M$ be the orthogonal frame bundle to the riemannian manifold $M$. This is an $O(n)$ bundle. Taking $V = \mathbb{R}^n$ with the inclusion $\rho : O(n) \to \text{GL}(V)$ returns the tangent bundle $T_*M$, that is

$$T_*M = O(T_*M) \times_\rho \mathbb{R}^n.$$ 

Taking $V = \text{Hom}(\mathbb{R}^n, \mathbb{R})$ and the dual representation returns the cotangent bundle $T^*M$. Taking $V$ to the action of $O(n)$ on the $p$-alternating multilinear maps $(\mathbb{R}^n)^p \to \mathbb{R}$ returns the bundle of $p$-forms, and so forth.

**Exercise 3.5:** Show that if $G = SO_3$, then the adjoint representation is isomorphic to the standard representation. (Hint: consider the vector space isomorphism $\mathbb{R}^3 \to so_3$ which takes 

$$(x, y, z) \mapsto \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}.$$ 

Since $SU_2$ is the double cover of $SO_3$, we conclude that the adjoint representation of $SU_2$ is the same as the composition of the 2-fold cover $SU_2 \to SO_3$ with the standard representation of $SO_3$ on $\mathbb{R}^3$. This fact is exploited many times in Fintushel and Stern’s work.

A useful observation is that if $E = P \times_\rho V$, then sections of $E$ are just equivariant maps $f \in \text{Maps}_G(P, V)$. Indeed, the correspondence is given by $(f : P \to V) \mapsto (\phi_f \in \Gamma(E))$ where $\phi_f(m) = \text{equivalence class of } (p, f(p))$ where $p$ is any point in $P$ which lies over $m$, so $p \in \pi^{-1}(m)$.

**Exercise 3.6:** Check that this map makes sense and is a bijection.

### 3.4 Characteristic Classes of Vector Bundles

We would like to define characteristic classes for a vector bundle to be the characteristic classes of the principal bundle from which it is constructed, but a moment’s thought will convince the reader that it is not always obvious which principal bundle a vector bundle comes from. For example, any $\mathbb{R}^3$ bundle (with a metric) over
${\bf S}^4$ can be thought of as either associated to an $SO_3$ bundle via the standard representation or to an $SU_2$ bundle via the adjoint representation. What should its characteristic classes be, Pontrjagin classes or Chern classes?

Given a $\mathbb{R}^k$ (resp. $\mathbb{C}^k$) vector bundle $E \to M$, the bundle of frames $GL(E)$ is a principal $GL_k \mathbb{R}$ (resp. $GL_k \mathbb{C}$) bundle so that $E$ is the vector bundle associated to it using the standard representation (see Example E in Section 3.1). If $E$ has a (Hermitean) metric, then $O(E)$ (resp. $U(E)$) will denote the principal $O_k$ (resp. $U_k$) bundle of orthogonal (resp. unitary) frames. We then define the characteristic classes of $E$ to be the characteristic classes of the principal bundle obtained this way. Plodding through the following exercise will illustrate the point.

**Exercise 3.7:** Let $P \to M$ be a principal $SU_2$ bundle over an oriented 4-manifold. Let $E \to M$ be the $\mathbb{C}^2$ bundle associated to $P$ via the standard representation and let $ad \ g \to M$ denote the $su(2) \cong \mathbb{R}^3$ bundle associated to $P$ via the adjoint representation. Then $c_2(E) = 4 \cdot p_1(ad \ g)$. (Hint: compute the map $BSU_2 \to BSO_3$ on integral cohomology.)

There are certain exceptions to this way of taking characteristic classes; for example, if someone refers to the Pontrjagin classes of a complex vector bundle, they mean the Pontrjagin classes of the underlying real oriented vector bundle.

## 4 Connections

There are many definitions of connections in principal bundles and vector bundles. One differential geometer I asked told me he knew 13 different definitions! He didn’t tell me all thirteen, so I have narrowed it down a bit. There are also local and global formulations, and each one is useful in certain contexts. I will give you three global definitions and a host of local formulas. By “global” I mean the definition does not use a choice of coordinates or a trivialization of a bundle. By “local” I mean a definition or formula which works in a trivialized bundle. We begin with the definition of a canonical family of vector fields on $P$.

### 4.1 The Fundamental Vector Fields
If \( \pi : P \rightarrow M \) is a principal \( G \) bundle, then for each \( p \in P \) we have the map \( i_p : G \rightarrow P \) given by the formula \( i_p(g) = p \cdot g \). Suppose we choose \( X \in g = \text{Lie algebra of } G = T_eG \). Then let \( X^* \) be the vector field on \( P \) whose value at \( p \in P \) is \( X^*_p = (i_p)_*(X) \). \( X^* \) is called the fundamental vector field corresponding to \( X \). This vector field is just the image of the left-invariant vector field on \( G \) corresponding to \( X \) under the embeddings of \( G \) in \( P \) given by \( g \mapsto p \cdot g \).

**Exercise 4.1:** \( X^* \) is smooth, and vertical, i.e \( \pi^* (X^*) = 0 \), where \( \pi^* : T^*_pP \rightarrow T^*_pM \).

The fundamental vector fields determine the “vertical” tangent directions in a principal bundle.

### 4.2 Connections in Principal Bundles

**Definition # 1: An equivariant Lie algebra valued 1-form on \( P \)**

Let \( P \rightarrow M \) be a smooth principal \( G \) bundle over \( M \). A connection on \( P \) is a Lie algebra valued 1-form \( A \) on \( P \) which satisfies:

(a) \( (i)_* A(X^*) = X \) for every \( X \in g \)

(b) \( R_g^*(A) = ad(g^{-1})A \) for each \( g \in G \).

The second condition says that if \( Z \) is a vector field on \( P \) and \( g \in G \), then \( A((R_g)_*(Z)) = ad(g^{-1}) \cdot A(Z) \), where \( R_g : P \rightarrow P \) denotes right translation by \( g \).

Locally, a 1-form on a manifold with values in \( g \) has the expression:

\[
(dz_1 \otimes A_1) + \cdots + (dz_k \otimes A_k)
\]

where the \( z_i \) are coordinates and the \( A_i \) are elements of \( g \). Globally, a 1-form on \( P \) with values in \( g \) is a section of the tensor product of two vector bundles \( \bigwedge^1 T^* P \otimes (P \times g) \), where \( P \times g \) is the trivial Lie algebra bundle over \( P \).

**Definition # 2: An Equivariant Horizontal Distribution on \( P \)**

A connection on a principal \( G \) bundle \( P \rightarrow M^n \) is an \( n \)-dimensional distribution on \( T^*_pP \), so \( H_p \subset T^*_pP \) is an \( n \)-dimensional subspace for each \( p \in P \), so that:

(a) \( H_p \subset T^*_pP \xrightarrow{\pi^*} T^*_p\pi(p)M \) is an isomorphism for each \( p \in P \)

(b) The distribution is equivariant, i.e. \( H_{pg} = (R_g)_*(H_p) \).

(c) \( H_p \) varies smoothly with \( p \).

The \( H_p \) are called the Horizontal Subspaces.
Theorem

The correspondence $A \mapsto \text{Kernel}A$ shows the two definitions to be equivalent.

We leave the obvious proof to the reader. Use the fact that the choice of $H_p$ decomposes $T_pP = H_p \oplus V_p$, where $V_p$ is the vertical subspace spanned by all the fundamental vector fields.

From definition # 2 we see that a connection on $P$ determines a unique way to lift tangent vectors from $M$: if $X_m \in T_m M$ let $\tilde{X}_p \in T_p P$ be the unique vector in $H_p$ lying over $X_m$, where $p \in \pi^{-1}(m)$. The lift $\tilde{X}$ is called the horizontal lift of $X$. Since we can lift vectors from $M$ to $P$, it is reasonable to ask whether we can lift pathes in such a way that the tangent vectors to the lifted path are the horizontal lifts of the tangent vectors to the path in $M$. This is indeed possible; see the next section.

Examples

A. If $P \cong M \times G$, the trivial bundle, then the trivialization gives a decomposition $T_{(m,g)}P = T_m M \oplus T_g G$. Thus a trivialization hands us a choice of horizontal and vertical subspaces, this choice is called the trivial connection (with respect to this trivialization). The corresponding $g$ valued 1-form on $P$ takes a vector $X = (X_1, X_2) \in T_m M \oplus T_g G$ to $(L_{g^{-1}})_*(X_2) \in T_e G = g$. A different trivialization of the bundle will yield a different trivial connection.

More generally, given any $g$-valued 1-form on $M$, say $A \in \Omega^1 \otimes g$, one gets a connection on $P$ whose value at $X = (X_1, X_2) \in T_m M \oplus T_g G$ is just $\text{ad}_{g^{-1}} A(X_1) + (L_{g^{-1}})_*(X_2)$. It is easy to check that the two defining conditions for a connection hold. Conversely, any connection on $P$ can be pulled back to a $g$-valued 1-form on $M$ using the section $m \mapsto (m, e)$ and equivariance implies:

$$\{ \text{connections on } P \} \cong \{ A \in \Omega^1 \otimes g \}.$$

It is important to remember that this correspondence depends on the choice of trivialization of the principal bundle $P$. The following important exercise shows a change in trivialization changes the connection.

Exercise 4.2: Choose a connection in the trivial bundle $P \longrightarrow M$. Trivialize $P$ by choosing a section $s : M \longrightarrow P$ so that the connection pulls back to the a 1-form $A \in \Omega^1 \otimes g$. Choose a map $h : M \longrightarrow G$ and use $h$ to define a new section by the
formula $s_h(m) = s(m) \cdot h(m)$. Then in this new trivialization the connection has the form

$$ad_{h^{-1}}A + (L_{h^{-1}})_* \cdot dh.$$ 

This means that if $X \in T_mM$, then evaluating the pullback of the connection using $s_h$ on $X$ yields $ad_{h(m)^{-1}}A(X) + (L_{h(m)^{-1}})_* \cdot dh(X)$. Notice that $dh(X) \in T_{h(m)}G$.

(Hint: use the product rule

$$\frac{d}{dt}|_{t=0} s(\alpha t) h(\alpha t) = s_\ast(\dot{\alpha}_0)h(\alpha_0) + (i_{s(m)})_\ast h_\ast(\dot{\alpha}_0)$$

for $\alpha$ a path with derivative $X$.)

B. Any principal bundle has a connection. This follows from Example A, the fact that $tA_0 + (1-t)A_1$ is a connection if $A_0$ and $A_1$ are, and an argument involving a partition of unity.

The collection of all connections on $P$ forms a topological space $\mathcal{A}$ which is a subset of the vector space of all Lie algebra valued 1-forms on $P$. However, $\mathcal{A}$ is not a linear subspace, since the sum of two connections is not a connection. It is, however, an affine space. In fact, a choice of a base connection determines an isomorphism of $\mathcal{A}$ with $\Omega^1(ad g)$. This can be seen in the following way. Fix a connection $A_0$. Recall from Exercise 3.6 that sections of $P \times_{\rho} V$ are the same as equivariant maps from $P$ to $V$. Let $A$ be another connection. Given a vector field $X$ on $M$, we can construct a function $\tilde{A}_X : P \rightarrow g$ by the formula:

$$\tilde{A}_X(p) = A(\tilde{X}_p),$$

where $\tilde{X}_p$ is the horizontal lift of $X$ to $T_pP$ with respect to $A_0$. The properties of a connection shows that $\tilde{A}_X$ is indeed an equivariant map from $P$ to $g$, and therefore $\tilde{A}$ determines an element of $\Omega^1(ad g)$.

Conversely, given $\tilde{A} \in \Omega^1(ad g)$ and a vector $Y \in T_pP$, decompose $Y = Y_1 + Y_2$ into its horizontal and vertical parts with respect to $A_0$. Then one can construct a connection $A$ on $P$ by taking $A(Y) = \tilde{A}(Y_1)(p) + Y_2$. Thus the choice of $A_0$ provides us with an identification $\mathcal{A} \approx \Omega^1(ad g)$.

**Exercise 4.3:** Recall that on a trivialized bundle $M \times G \rightarrow M$, we have a natural trivial connection. Also $\Omega^1(ad g)$ is naturally isomorphic to $\Omega^1 \otimes g$. Finally we have seen in Example A that pulling back the connection on $P$ using the section
gives an isomorphism $A \approx \Omega^1 \otimes g$. Show that this identification is the same as the one given by the procedure described in the preceding paragraphs.

4.3 Holonomy

Let $A$ be a connection in a principal bundle $P \rightarrow M$. Let $\alpha : I \rightarrow M$ be a smooth path and $p_0 \in P$ a point lying over the initial point of $\alpha$.

**Proposition** There exists a unique lifting $\beta : I \rightarrow P$ so that for each $t \in I$, the derivative of $\beta$ at $t$ is the horizontal lift of the derivative of $\alpha$ at $t$. The lift $\beta$ is called the horizontal lift of $\alpha$.

**Proof.** By since a principal bundle is a fiber bundle, there exists some lift, say $\gamma$. Any other lift is of the form $\beta_t = \gamma_t \cdot g_t$ for some path $g : I \rightarrow G$ with $g(0) = 1$. Now $\beta$ is the horizontal lift if and only if $A(\dot{\beta}_t) = 0$ for all $t \in I$.

We have:

$$A(\dot{\beta}_t) = A(\dot{\gamma}_t g_t) = A(-\dot{\gamma}_t g_t + \dot{\gamma}_t \dot{g}_t) \text{ (Product rule)}$$

$$= ad_{g_t} A(\dot{\gamma}_t) + g_t^{-1} \dot{g}_t g_t$$

Thus $\beta$ is a horizontal lift if and only if

$$A(\dot{\gamma}_t) = -\dot{g}_t g_t^{-1}.$$

To find the appropriate $\gamma : I \rightarrow G$ is just a matter of solving an ordinary differential equation in $G$; this follows from the standard:

**Fact** Given a map $X : I \rightarrow g = T_1 G$, then there is a unique solution $g : I \rightarrow G$ satisfying $g_0 = 1$ and $-\dot{g}_t^{-1} \dot{g}_t = X_t$.

□

4.4 Corollary: Any connection $A$ defines a map $H_A : \text{Loops}(M,m_0) \rightarrow G$ called the Holonomy map.

This holonomy map does not define a map on $\pi_1$, unless the connection is Flat, a term we will define later.

Actually, one can recapture the connection by knowing what the horizontal lifts of all pathes are, just define the horizontal subspace at $p \in P$ to be the space spanned by the derivatives of the horizontal lifts of all pathes through $\pi(p)$. One
can therefore give yet another definition of a connection in a principal bundle to be a coherent way to lift paths in $M$ so that equivariance is satisfied.

### 4.5 Connections in Vector Bundles

We now turn to connections in vector bundles. One should think of these as a way to differentiate sections of vector bundles in the direction of a vector field on the base manifold. For example, given a function $f : \mathbb{R}^n \to \mathbb{R}$, and a vector field $X$ on $\mathbb{R}^n$, then we can take the derivative of $f$ along the vector field to get another function $df_X : \mathbb{R}^n \to \mathbb{R}$. Similarly, if we have a (vector valued) function $f : \mathbb{R}^n \to \mathbb{R}^p$, and a vector field $X$ on $\mathbb{R}^n$ we can apply the preceding procedure componentwise to obtain a new function $df_X : \mathbb{R}^n \to \mathbb{R}^p$. Again this can be done replacing $\mathbb{R}^n$ by any manifold $M$, thus there is a natural way to take derivatives of functions $M \to \mathbb{R}^p$ along vector fields. But functions are just sections of trivial bundles, and the general notion of a connection in a vector bundle is a way to do this when the bundle is not trivial.

Let $E \to M$ be a vector bundle. $\Omega^p(E)$ will denote the $p$-forms with values in $E$, i.e. smooth sections of the vector bundle:

$$\bigwedge^p T^*M \otimes E \to M.$$ 

The symbol “$\otimes$” means the tensor product of the two bundles. Thus both $\bigwedge^p T^*M$ and $E$ contribute to the “twisting” of the bundle $\bigwedge^p T^*M \otimes E \to M$.

Notice that $\Omega^0(E)$ is isomorphic to $\Gamma(E)$ via the map $f \otimes \phi \mapsto f\phi$, since $\bigwedge^0 T^*M$ is just the trivial line bundle.

**Definition # 3 Affine Connection**

An **affine connection**, also called a **covariant derivative** on $E$ is a linear mapping

$$\nabla : \Omega^0(E) \to \Omega^1(E)$$

satisfying the Leibniz rule:

$$\nabla(f\phi) = df \otimes \phi + f\nabla\phi$$

for $f \in C^\infty(M, \mathbb{R})$ and $\phi \in \Omega^0(E)$.

If $X$ is a vector field on $M$, then we will write $\nabla_X(\phi)$ for the evaluation $(\nabla\phi)(X)$.

If $E$ is associated to a principal $U_n$ or $SO_n$ bundle via the standard representation, then $E$ has a fiberwise inner product $(\ , \ )$. We say $\nabla$ is **compatible with the**
metric if
\[ d(\phi, \theta) = (\nabla \phi, \theta) + (\phi, \nabla \theta) \]

This expression makes sense since \((\phi, \theta)\) is a smooth function on \(M\), thus \(d(\phi, \theta) \in \Omega^1\). Also \((\nabla \phi, \theta) \in \Omega^1\) since we can evaluate it on a vector field \(X: (\nabla \phi, \theta)(X) = (\nabla_X \phi, \theta)\).

Compatibility with a metric refers to the metric on the vector bundle, not on the Riemannian metric on \(M\).

Examples

A. Let \(E = M \times \mathbb{C}^m\), the trivial \(\mathbb{C}^m\) bundle. Sections of \(E\) are just functions \(\phi : M \to \mathbb{C}^m\). Let \(d : \Omega^0 \to \Omega^1\) denote the exterior derivative. It is defined for complex valued functions on \(M\). Then \(d\) defines the trivial connection on \(E\):

\[ d : \Omega^0(E) \to \Omega^1(E) \]

\[ \phi = (\phi_1, \ldots, \phi_m) \mapsto (d\phi_1, \ldots, d\phi_m). \]

Using this example and a partition of unity we see that any vector bundle admits an affine connection.

B. Again let \(E = M \times \mathbb{C}^m\), and suppose that \(A \in \Omega^1 \otimes g\) is a 1-form on \(M\) with values in the Lie algebra \(g\) of \(G\), where \(G\) is some subgroup of \(GL_m(\mathbb{C})\). Let

\[ d^A : \Omega^0(E) \to \Omega^1(E) \]

be given by the formula:

\[ \phi = (\phi_1, \ldots, \phi_m) \mapsto (d\phi_1, \ldots, d\phi_m) + A\phi \]

The term \(A\phi\) is a 1-form with values in \(E\), since given a vector field \(X, A(X) \in g\) which acts on sections of \(E\) by left matrix multiplication.

**Exercise 4.4:** Show that \(d^A\) is an affine connection on \(E\). If \(G = U_m\), show that \(d^A\) is compatible with the (trivial) Hermitian metric on \(E\) if and only if \(A \in \Omega^1(u_m)\).

We will now show that any connection on \(E\) can be written locally as \(d + A\) for some Lie algebra valued 1-form \(A\) on \(M\).

A convenient way to show this is to use local frames. Let \(\{e_i\}\) be a local frame for a vector bundle \(E\). We assume \(E\) is a \(GL_m\) vector bundle. Let \(\nabla\) be a covariant derivative and let \(A_{r,s}\) be the matrix of 1-forms determined by the expression:

\[ \nabla e_s = \sum_r A_{r,s} \otimes e_r. \]
(If $E$ is an $O_m$ or $U_m$ bundle and the $e_i$ are orthogonal, then $A_{r,s}$ is an $O_n$ or a $U_n$ valued form if $\nabla$ is compatible with the metric.) Now if $\phi \in \Gamma(E)$, we can write $\phi = \Sigma \phi_i e_i$ and so we must have:

\[
\nabla \phi = \Sigma_r (d\phi_r \otimes e_r + \phi_r \Sigma_s A_{s,r} \otimes e_s)
\]

\[
= \Sigma_r (d\phi_r + \Sigma_s \phi_s A_{r,s}) \otimes e_r
\]

Writing $\phi$ as a column vector, we can write this as

\[
\nabla \phi = (d + A)\phi.
\]

Hence in local coordinates any connection is of the form $d + A$.

This example hints (strongly) at the relationship between Definitions 1 and 3. We saw that in a trivial principal bundle, a connection is determined by a Lie algebra valued 1-form on $M$. The same is true in the preceding example. One sees a local equivalence of the two notions of connections in principal bundles and connections in vector bundles. In the next section we will give a global identification of these two notions, without any reference to a trivialization of the bundle.

C. The **Fundamental Theorem of Riemannian Geometry** states that every Riemannian manifold possesses a unique connection on its tangent bundle which satisfies two conditions (compatible with the metric and torsion free). Thus invariants of Riemannian manifolds can be constructed by working with this connection called the *Riemannian or Levi-Civita* connection. It should be stressed that the Riemannian connection depends only on the metric. Textbooks on Differential Geometry contain expressions for the Matrix valued 1-form of the Riemannian Connection this connection in terms of the $g_{i,j}$ and their derivatives.

### 4.6 Equivalence of the Two Notions of Connection

Let $E = P \times_p V \to M$ be a vector bundle associated to a principal bundle $P \to M$. We saw above that the sections of $E$ are equivariant maps $\phi \in Maps_G(P,V)$. Notice that for a vector space $V$, the tangent space $T_v V$ is canonically identified with $V$ itself.

Let $A$ be a connection in the principal bundle $P \to M$; we think of $A$ as a horizontal distribution. Given a section $\phi : P \to V$ and a vector field $X$ on $M$, we want to construct another section $\nabla^A_X \phi : P \to V$. Given $p \in P$ let $\tilde{X}_p$ be the
unique horizontal lift of $X_{\pi(p)}$ to $T_p P$ given by the connection. Define $(\nabla^A_X(\phi))(p)$ to be $d\phi(\tilde{X}_p) \in T_v V = V$.

**Theorem**

This construction defines a bijection between connections in principal $U_m$ or $O_m$ bundles and affine connections in $U_m$ or $O_m$ vector bundles compatible with the metric.

Path lifting in a principal bundle corresponds to *Parallel transport* of frames in a vector bundle. If I choose a frame in the fiber of $E$ over $m \in M$ it is easy to see that lifting a path $\alpha_t$ starting at $m$ to $P$ gives me a way to choose a frame in the fiber over each point of the path, i.e. it hands me a trivialization of the pullback bundle $\alpha^*(E)$ over $I$. This is one of the older definitions of a connection, the “répere mobile” of E. Cartan.

**4.7 Extending covariant derivatives**

Suppose $A$ is a connection in a principal bundle, we saw how to define an affine connection $\nabla^A : \Omega^0(E) \to \Omega^1(E)$. This can be extended to

$$d^A : \Omega^k(E) \to \Omega^{k+1}(E)$$

by forcing the Leibniz rule to hold; so

$$d^A(\omega \otimes \phi) = d\omega \otimes \phi + (-1)^k \omega \wedge \nabla^A \phi.$$ 

**Notation** $\nabla^A$ and $d^A$ are used interchangably.

The resulting sequence:

$$\Omega^{k-1}(E) \xrightarrow{d^A} \Omega^k(E) \xrightarrow{d^A} \Omega^{k+1}(E)$$

Need not be a complex. In fact, we will see below that this sequence is a complex iff $A$ is a *flat* connection.

There are two special cases of this which are particularly important, namely the bundle associated to $P$ via the standard representation and the adjoint bundle. If $E$ is the bundle associated to $P$ via the standard representation (where $G$ is a subgroup of $GL_k(\mathbb{R})$ or $GL_k(\mathbb{C})$) then the adjoint bundle is a sub-vector bundle of the bundle $\text{Hom}(E, E)$, since $\text{Hom}(E, E) = P \times_{ad} gl_k$. If you choose a local basis of sections $\{e_i\}$ of $E$, then the covariant derivative on $E$ has the form $d^A = d + A$ for some $A \in \Omega^1 \otimes gl_k$, as explained in 4.5.
Exercise 4.5: (1) Show that the induced connection $d^A : \Omega^1(E) \to \Omega^2(E)$ is given by the formula:

$$\phi \mapsto d\phi + A \wedge \phi$$

where $A \wedge \phi$ means the following. Wedging gives a map

$$\Omega^1(Hom(E,E) \times \Omega^1(E) \to \Omega^2(Hom(E,E) \otimes E).$$

We compose this with the natural evaluation map $Hom(E,E) \otimes E \to E$.

(2) Show that the induced connection $d^A : \Omega^1(Hom(E,E)) \to \Omega^2(Hom(E,E))$ is given by the formula:

$$B \mapsto dB + A \wedge B.$$ 

This time $A \wedge B$ means the composition of the exterior product map

$$\Omega^1(Hom(E,E)) \times \Omega^1(Hom(E,E)) \to \Omega^2(Hom(E,E) \otimes Hom(E,E))$$

with the Lie bracket $[ , ] : Hom(E,E) \otimes Hom(E,E) \to Hom(E,E)$. (Check that this map makes sense globally.)

5 Curvature

5.1 Definition of Curvature

Let $A$ be a connection, $d^A$ its associated covariant derivative in the associated bundle $E = P \times \rho V$.

Theorem

The composition

$$\Omega^0(E) \xrightarrow{d^A} \Omega^1(E) \xrightarrow{d^A} \Omega^2(E)$$

is a zeroth order operator. More precisely, there exists an $F^A \in \Omega^2(Hom(E,E))$ such that $d^A d^A \phi = F^A \cdot \phi$.

$F^A$ is called the Curvature of the connection $A$, and it actually lies in the subspace $\Omega^2(ad g) \subset \Omega^2(Hom(E,E))$.

Proof.
We give a local proof. Choose a local frame \( \{ e_i \} \). In this frame the connection has the form \( d^A = d + A \). Let \( \phi \) be a section; write it as a column vector. Then

\[
d^A d^A \phi = d^A (d\phi + A\phi)
\]

\[
= d^2 \phi + A \wedge d\phi + d(A\phi) + A \wedge A\phi
\]

\[
= A \wedge d\phi + dA \wedge \phi - A \wedge d\phi + A \wedge A\phi
\]

\[
= (dA + A \wedge A)\phi
\]

The minus sign comes from the fact that the entries of \( A \) are 1-forms and \( d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta \) for \( \omega \) a p-form.

\[
\square
\]

**Exercise 5.1:**

1. What does \( A \wedge A \) mean?

2. Redo this proof in an even more local version, i.e. Let \((x_1, \cdots, x_n)\) be coordinates on \( M \), \((e_1, \cdots, e_k)\) a local frame for the bundle \( E \), \( A = A_{i,j} dx_1 + \cdots + A_{n,j} dx_n \), where \( \nabla e_i = \sum A_{i,j} \otimes e_j \). Use the definition of the extended \( d^A \) given in section 4.7. This exercise will help you understand the meaning of the expression \( d^A = d + A \), as well as give you some experience in working with differential forms.

3. Give a global version of this proof as follows:

   a. Show that \( d^A d^A (f\phi) = f d^A d^A (\phi) \) for \( f \in C^\infty (M) \) and \( \phi \in \Omega^p (E) \)

   b. Let \( L : \Omega^0 (E) \rightarrow \Omega^2 (E) \) be any \( C^\infty (M) \)-linear map. Then there is an \( \hat{L} \in \Omega^2 (\text{Hom} (E, E)) \) such that \( L(\phi) = \hat{L} \circ \phi \) for all \( \phi \in \Omega^0 (E) \). The expression \( \hat{L} \circ \phi \) means the section of \( E \) which is the composite of \( \phi \) and the bundle homomorphism \( \hat{L} \).

We have also obtained a local expression for \( F^A \): in a local frame

\[
F^A = dA + A \wedge A.
\]

Notice that \( A \wedge A \) is a combination of matrix multiplication and wedge product. Thus the entries of \( F^A \) are

\[
F_{r,s} = dA_{r,s} + \sum_k A_{r,k} \wedge A_{k,s}.
\]

**5.2 Wedge Products**

At this point it is convenient to clarify some notation which arises from time to time, especially in the next section. Suppose we are given two differential forms
A ∈ Ω^p(E) and B ∈ Ω^q(E). Then the expression \( A \wedge B \) a priori takes its value in \( \Omega^{p+q}(E \otimes E) \). But if we are given a bilinear form \( E_m \times E_m \to F_m \), where \( F \) is some vector bundle over \( M \) and \( E_m, F_m \) denotes the fibers over \( M \), then using this form we can define the product \( A \wedge B \in \Omega^{p+q}(F) \). A typical example is when \( E = ad g \) is the Lie algebra bundle of some principal bundle. Then you can take the Lie bracket for the bilinear form, so that \( A \wedge B \in \Omega^{p+q}(ad g) \). Another possibility is to identify \( G \) with a matrix group and take ordinary matrix multiplication, so that \( A \wedge B \in \Omega^{p+q}(gl_k) \).

**Exercise 5.2:** Show that if \( \wedge_1 : \Omega^1(ad g) \times \Omega^1(ad G) \to \Omega^2(ad g) \) is given by matrix multiplication, and if \( \wedge_2 \) is given by taking \( \frac{1}{2} \) Lie bracket, then \( A \wedge_1 A = A \wedge_2 A \), but \( A \wedge_1 B \neq A \wedge_2 B \) in general.

In the following section we will see expressions like \( Tr(F^A \wedge F^A) \). This means first wedge the two forms together to get a form \( F^A \wedge F^A \in \Omega^4(ad g \otimes ad g) \), then take the fiberwise trace \( Tr : g \otimes g \to C \) to get a form \( Tr(F^A \wedge F^A) \in \Omega^4 \).

**Exercise 5.3:** Show that if \( f : g \times \cdots \times g \to C \) is a multilinear function which is \( ad \)-invariant, i.e \( f(x_1, \cdots, x_k) = f(gx_1g^{-1}, \cdots, gx_kg^{-1}) \) for each \( g \in G \), then \( f \) determines a multilinear map
\[
\Omega^p(ad g \otimes ad g) \to \Omega^p.
\]

### 5.3 Flat Connections and Twisted Cohomology

**Definition** A connection \( A \) is **flat** if \( F^A = 0 \) pointwise.

**Example** If \( A \) is flat, then the sequence:
\[
\cdots \to \Omega^{k-1}(E) \xrightarrow{d^A} \Omega^k(E) \xrightarrow{d^A} \Omega^{k+1}(E) \to \cdots
\]
is a complex, since \( d^A d^A = F^A = 0 \). The following theorem shows that the cohomology of this complex is a familiar one.

**Theorem**

If \( A \) is flat, then the holonomy representation
\[
\text{Loops}(M, m_0) \to G
\]
is well defined on the fundamental group, and furthermore, the homology of the chain complex described above is just the homology of $M$ with local coefficients in $V$ given by the composite:

$$\pi_1(M, m_0) \longrightarrow G \longrightarrow GL(V).$$

\[\Box\]

5.4 Self-Dual Connections on a 4-Manifold

Another important type of connection is a self-dual connection on a 4-manifold.

Let $(M^4, g)$ be an oriented 4-manifold. We call a connection $A$ on $M$ **self dual** if its curvature form is self-dual; i.e. if

$$F^A = *F^A$$

where $*$ is the Hodge $*$ operator $*\Omega^2(ad\ g) \longrightarrow \Omega^2(ad\ g)$. We gave the definition of the Hodge $*$ operator on regular $p$-forms, but the definition extends to $p$-forms with values in any bundle: just apply $*$ to the form part and leave the coefficients alone.

If $A$ is a self dual connection, then the sequence:

$$0 \longrightarrow \Omega^0(ad\ g) \xrightarrow{d^A} \Omega^1(ad\ g) \xrightarrow{P_-d^A} \Omega^2(ad\ g) \longrightarrow 0$$

is a complex, since $P_-d^A d^A = P_- F^A = 0$. This complex is called the **Fundamental Elliptic Complex**. Its importance lies in the fact that its first cohomology can be identified with the tangent space to the moduli space of self dual connections on a four manifold. See Atiyah-Hitchin-Singer [AHS].

5.5 The Chern-Weil Description of Characteristic Classes

The Chern-Weil theory is a means of describing characteristic classes of vector bundles using Differential Geometry, instead of the topological method of pulling back universal cohomology classes. It says that Chern classes and Pontrjagin classes of a vector bundle are represented in DeRham theory by differential forms which are functions of the curvature of a connection in the bundle. For a good and quick introduction to this read Appendix C of the beautiful book *Characteristic Classes* by Milnor and Stasheff.
Definition an invariant polynomial on \( M_{m,m}(\mathbb{C}) \) is a function \( p : M_{m,m}(\mathbb{C}) \rightarrow \mathbb{C} \) so that \( p(TXT^{-1}) = p(X) \) for each invertible matrix \( T \), and so that \( p \) is expressible as a polynomial in the entries of the matrix.

For example, the trace and the determinant are invariant polynomials.

Exercise 5.4: If \( p \) is an invariant polynomial and \( \omega \in \Omega^2(\text{Hom}(E, E)) \) then \( p(\omega) \in \Omega^2 \) is a well-defined differential form.

Now let \( P \rightarrow M \) be a principal \( U_m \) bundle. Choose a connection \( A \) and let \( F^A \) be its curvature.

Theorem

For any invariant polynomial \( p \), \( p(F^A) \) is a closed 2 form and its cohomology class is independent of the connection \( A \).

\[ \square \]

All invariant polynomials are products of the elementary symmetric polynomials \( \sigma_k \) defined by:

\[
\begin{align*}
\sigma_1(X) &= a_1 + \cdots + a_n \\
\sigma_2(X) &= \sum_{i<j} a_i a_j \\
&\vdots \\
\sigma_n(X) &= a_1 \cdots a_n
\end{align*}
\]

where the \( a_i \) are the eigenvalues of \( X \).

Recall that \( H^*(BU_m, \mathbb{C}) = \mathbb{C}[c_1, \ldots, c_m] \), with each Chern class \( C_k \) of dimension \( 2k \). Thus for any \( U_m \) bundle \( E \rightarrow M \) we have the Chern classes \( c_k(E) \in H^{2k}(M; \mathbb{C}) \).

Theorem

If \( A \) is any connection in \( E \) and \( F^A \) its curvature, then \( c_k(E) \) is represented by the DeRham class:

\[
\frac{1}{(2\pi i)^k} \sigma_k(F^A).
\]

\[ \square \]

There is a similar formulation for the Pontrjagin classes.

Example: Let \( E \rightarrow M \) be an \( SU_2 \) bundle. Then \( \sigma_1 : M_{2,2}(\mathbb{C}) \rightarrow \mathbb{C} \) is the determinant. If \( A \) is an \( SU_2 \) connection, then \( F^A \in \Omega^2(\text{ad } su_2) \). For elements \( X \in su_2 \), \( \det(X) = 1/2 \cdot \text{Tr}(X^2) \). Thus on a 4 manifold \( M \),

\[
c_2(E)(M) = \frac{1}{4\pi^2} \int_M \det(F^A) = \frac{1}{8\pi^2} \int_M \text{Tr}(F^A \wedge F^A).
\]
A similar computation shows that on an $SO_3$ bundle $E$ over a 4 manifold $M$,

$$p_1(E)[M] = \frac{1}{8\pi^2} \int_M Tr(F^A \wedge F^A).$$

5.6 The Yang-Mills Functional

Let $E\rightarrow M$ be an $SU_2$-vector bundle over a 4-manifold. There is a natural (pointwise) inner product on 2-forms with values in the adjoint bundle obtained by multiplying the Riemannian inner product on $\bigwedge^2 T^*M$ with the inner product on $su_2$: $Tr: su_2 \times su_2 \rightarrow \mathbb{R}$ given by $Tr(a, b) = \text{trace}(ab)$. Explicitly,

$$\langle A, B \rangle d\text{vol} = Tr(A \wedge *B).$$

We can now define the “strength” or “energy” of a connection $A$ on $E$ to be the integral of the pointwise norm (squared) of its curvature; this is called the Yang Mills functional of a connection.

$$\mathcal{YM}(A) = \int_M \|F^A\|^2 = \int_M Tr(F^A \wedge *F^A).$$

We can decompose $F^A$ into its self-dual and anti self-dual parts $F^A = F^A_+ + F^A_-$. Recall from Section 2.4 that this is a orthogonal splitting. Thus:

$$\mathcal{YM}(A) = \int_M Tr(F^A_+ \wedge *F^A_-) - Tr(F^A_- \wedge *F^A_+) \leq \int_M Tr(F^A_+ \wedge F^A_+) + Tr(F^A_- \wedge F^A_-)$$

with equality if and only if $F^A_- = 0$, i.e. $A$ is a self-dual connection. Furthermore, if $A$ is self-dual, then from the results of the previous section we see that $\mathcal{YM}(A) = -8\pi^2 c_2(E)[M]$. Since $\mathcal{YM}(A)$ is always positive, this forces $c_2(E)[M]$ to be negative. Put another way, there are no self-dual connections on a bundle $E$ with $c_2(E)[M]$ positive. Another consequence is that for any connection $A$ on $E$, $\mathcal{YM}(A) \leq -8\pi^2 c_2(E)[M]$.

6 Differential Operators

Let $E\rightarrow M$ be a vector bundle with fiber $\mathbb{R}^k$ and let $F\rightarrow M$ be a vector bundle with fiber $\mathbb{R}^l$. Then a Differential Operator from $E$ to $F$ is an $\mathbb{R}$ linear map

$$D: \Gamma(E)\rightarrow \Gamma(F)$$
such that locally $D$ is a differential operator from $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^k)$ to $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^l)$. This means that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$, then $Df : \mathbb{R}^n \rightarrow \mathbb{R}^k$ has the expression:

$$[Df]_i = \sum_{j=1,|I|}^{k} a_{i,j}^I D^I f_j.$$ 

The $I$ are multiindices; $I = (i_1, \cdots, i_k)$ and $D^I f$ means $\frac{\partial^{i_1 + \cdots + i_k} f}{\partial^{i_1} x_1 \cdots \partial^{i_k} x_k}$.

The order of a differential operator is the maximum $i_1 + \cdots + i_k$. 