quantitative information whether the relation between actions and payoffs is one of strategic substitutes or strategic complements. The businessman then knows whether, for example, he should try to be a first mover or a second mover, and whether he should keep his action secret or proclaim his action to the entire world.

We will explore this further in Chapter 14. For now, though, try to think about how you would model situations like these (noting that there is no universally right answer for any of them):

1. Two firms are choosing their research and development budgets. Are the budgets strategic complements or strategic substitutes?
2. Smith and Jones are both trying to be elected President of the United States. Each must decide how much he will spend on advertising in California. Are the advertising budgets strategic complements or strategic substitutes?
3. Seven firms are each deciding whether to make their products more special, or more suited to the average consumer. Is the degree of specialness a strategic complement or a strategic substitute?
4. Iran and Iraq are each deciding whether to make their armies larger or smaller. Is army size a strategic complement or a strategic substitute?

3.7 Existence of Equilibrium

One of the strong points of Nash equilibria is that they exist in practically every game one is likely to encounter. There are four common reasons why an equilibrium might not exist or might only exist in mixed strategies.

(1) An unbounded strategy space.

Suppose in a stock market game that Smith can borrow money and buy as many shares x of stock as he likes, so his strategy set, the amount of stock he can buy, is $[0, \infty)$, a set which is unbounded above. (Note, by the way, that we thus assume that he can buy fractional shares, e.g. $x = 13.4$, but cannot sell short, e.g. $x = -100$.)

If Smith knows that the price is lower today than it will be tomorrow, his payoff function will be $\pi(x) = x$ and he will want to buy an infinite number of shares, which is not an equilibrium purchase. If the amount he buys is restricted to be less than or equal to 1,000, however, then the strategy set is bounded (by 1,000), and an equilibrium exists– $x =, 000$.

Sometimes, as in the Cournot Game discussed earlier in this chapter, the unboundedness of the strategy sets does not matter because the optimum is an interior solution. In other games, though, it is important, not just to get a determinate solution but because the real world is a rather bounded place. The solar system is finite in size, as is the amount of human time past and future.

(2) An open strategy space. Again consider Smith. Let his strategy be $x \in [0, 1,000)$, which is the same as saying that $0 \leq x < 1,000$, and his payoff function be $\pi(x) = x$. Smith's strategy set is bounded (by 0 and 1,000), but it is open rather than closed, because he can choose any number less than 1,000, but not 1,000 itself. This means no equilibrium will exist, because he wants to buy 999.999... shares. This is just a technical problem; we ought to have specified Smith strategy space to be $[0, 1,000]$, and then an equilibrium would exist, at $x = 1,000$.

(3) A discrete strategy space (or, more generally, a nonconvex strategy space). Suppose we start with an arbitrary pair of strategies s_1 and s_2 for two players. If the players' strategies are strategic complements, then if player 1 increases his strategy in response to s_2, then player 2 will increase his strategy in response to that. An equilibrium will occur where the players run into diminishing returns or increasing costs, or where they hit the upper bounds of their strategy sets. If, on the other hand, the strategies are strategic substitutes, then if player 1 increases his strategy in response to s_2, player 2 will in turn want to reduce his strategy. If the strategy spaces are continuous, this can lead to an equilibrium, but if they are discrete, player 2 cannot reduce his strategy just a little bit– he has to jump down a discrete level. That could then induce Player 1 to increase his strategy by a discrete amount. This jumping of responses can be never-ending–there is no equilibrium.

That is what is happening in The Welfare Game of Table 1 in this chapter. No compromise is possible between a little aid and no aid, or between working and not working– until we introduce mixed strategies. That allows for each player to choose a continuous amount of his strategy.

This problem is not limited to games such as 2-by-2 games that have discrete strategy spaces. Rather, it is a problem of "gaps" in the strategy space. Suppose we had a game in which the Government was not limited to amount 0 or 100 of aid, but could choose any amount in the space $\{[0, 10], [90, 100]\}$. That is a continuous, closed, and bounded strategy space, but it is non-convex– there is gap in it. (For a space $\{x\}$ to be convex, it must be true that if x_1 and x_2 are in the space, so is $\theta x_1 + (1 - \theta)x_2$ for any $\theta \in [0, 1]$.) Without mixed strategies, an equilibrium to the game might well not exist.

(4) A discontinuous reaction function arising from nonconcave or discontinuous payoff functions.

Even if the strategy spaces are closed, bounded, and convex, a problem remains. For a Nash equilibrium to exist, we need for the reaction functions of the players to intersect. If the reaction functions are discontinuous, they might not intersect.

Figure 6 shows this for a two-player game in which each player chooses a strategy from the interval between 0 and 1. Player 1's reaction function, $s_1(s_2)$, must pick one or more value of s_1 for each possible value of s_2, so it must cross from the bottom to the top of the diagram. Player 2's reaction function, $s_2(s_1)$, must pick one or more value of s_2 for each possible value of s_1, so it must cross from the left to the right of the diagram. If the strategy sets were unbounded or open, the reaction functions might not exist, but that is not a problem here: they do exist. And in Panel (a) a Nash equilibrium exists, at the point, E, where the two reaction functions intersect.

In Panel (b), however, no Nash equilibrium exists. The problem is that Firm 2's reaction function $s_2(s_1)$ is discontinuous at the point $s_1 = 0.5$. It jumps down from $s_2(0.5) = 0.6$ to $s_2(0.50001) = 0.4$. As a result, the reaction curves never intersect, and no equilibrium exists.

If the two players can use mixed strategies, then an equilibrium will exist even for the game in Panel (b), though I will not prove that here. I would, however, like to say why it is that the reaction function might be discontinuous. A player's reaction functions, remember, is derived by maximizing his payoff as a function of his own strategy given the strategies of the other players.

Thus, a first reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is discontinuous in either his own or the other players' strategies. This is what happens in the Hotelling Pricing Game, where if Player 1's price drops enough (or Player 2's price rises high enough), all of Player 2's customers suddenly rush to Player 1.

A second reason why Player 1's reaction function might be discontinuous in the other players' strategies is that his payoff function is not concave. The intuition is that if an objective function is not concave, then there might be a number of maxima that are local but not global, and as the parameters change, which maximum is the global one can suddenly change. This means that the reaction function will suddenly jump from one maximizing choice to another one that is far-distant, rather than smoothly changing as it would in a more nicely behaved problem.

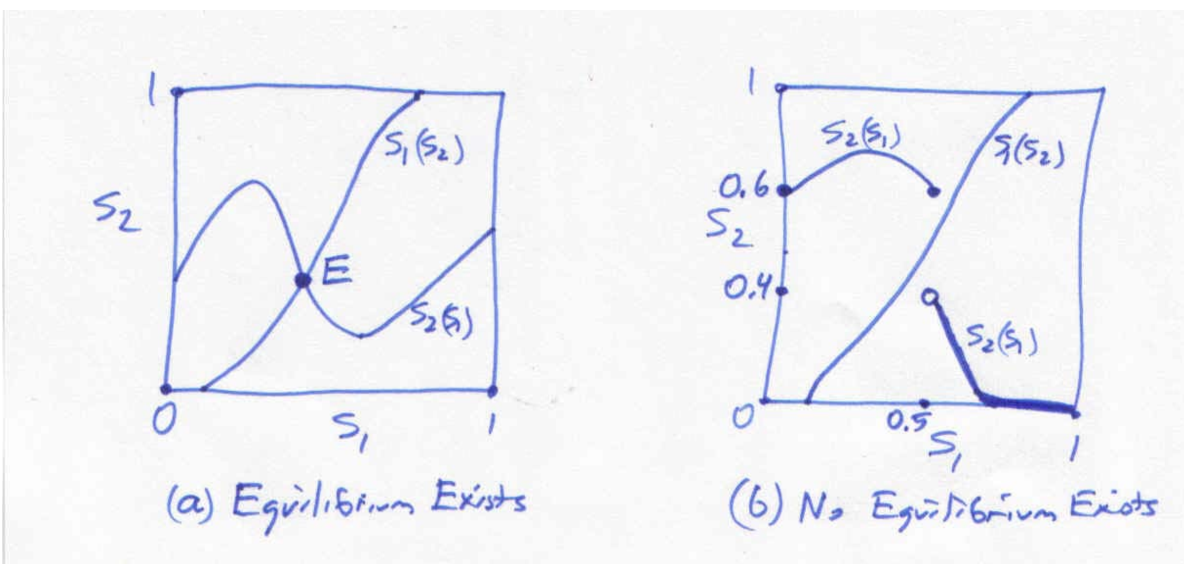

Figure 6: Continuous and Discontinuous Reaction Functions

Problems (1) and (2) are really decision theory problems, not game theory problems, because unboundedness and openness lead to nonexistence of the solution to a one-player maximization problem. Problems (3) and (4) are special to game theory. They arise because although each player has a best response to the other players, no profile of best choices is such that everybody has chosen his best response to everybody else. They are similar to the decision theory problem of nonexistence of an interior solution, but if only one player were involved, we would at least have a corner solution.

xxx ADD SOME NICE PARAGRAPH CONCLUDING FOR THE CHAPTER